Вычисление площади поверхности тела вращения

ДСК

$y = f(x) \in C([a;b])$

$$\boxed{Q_{0x} = 2\pi \int_a^b y \underbrace{\sqrt{1+(y'_x)^2}\, dx}_{dl}} \quad (1)$$

$x = f(y) \in C([c,d])$

$$\boxed{Q_{0y} = 2\pi \int_c^d x \underbrace{\sqrt{1+(x'_y)^2}\, dy}_{dl}} \quad (2)$$

ПЗФ

$$\begin{cases} x = x(t), \\ y = y(t) \end{cases}$$

$t \in [t_1, t_2]$

$$\begin{cases} x(t_1) = a, \\ x(t_2) = b \end{cases}$$

$$\begin{cases} y(t_1) = c, \\ y(t_2) = d \end{cases}$$

$$(1') \quad \boxed{Q_{0x} = 2\pi \int_{t_1}^{t_2} y(t) \underbrace{\sqrt{(x'_t)^2+(y'_t)^2}\, dt}_{dl}}$$

$$(2') \quad \boxed{Q_{0y} = 2\pi \int_{t_1}^{t_2} x(t) \underbrace{\sqrt{(x'_t)^2+(y'_t)^2}\, dt}_{dl}}$$

ПСК

$r = r(\varphi)$

$\alpha \le \varphi \le \beta$

$$(3) \quad \boxed{Q_{0\varphi} = 2\pi \int_\alpha^\beta r \sin\varphi \underbrace{\sqrt{r^2+(r'_\varphi)^2}\, d\varphi}_{dl}}$$

Площадь поверхности тела вращения – это внешняя оболочка (т.е сфера, а не весь шар).

Для того, чтобы не путать с площадью плоской фигуры введем обозначение Q.

Задачи: Ауд

[Е.Д]

6.519 а, б

6.529.

[ВОС] стр. 282

N 143

Д/З

[ВОС] стр. 282 – 285

N 137 а, б

151

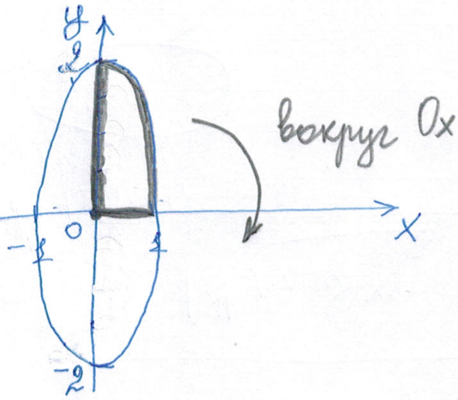
+ Д3 N 1

задача 3

[Е.Д] 6.519

а) $4x^2+y^2=4$ $\quad \boxed{x^2+\frac{y^2}{4}=1}$ – эллипс

Q_{ox} – ?



При вращении эллипса вокруг оси Ox получаем эллипсоид.

$x \in [-1, 1]$

Т.к фигура симметричная, то можно вычислить Q_{ox} при $x \in [0, 1]$, а потом умножить на два.

$$x^2+\frac{y^2}{4}=1$$

$$\frac{y^2}{4}=1-x^2$$

$$y^2=4(1-x^2)$$

$$y=\pm 2\sqrt{1-x^2}$$

Пусть $y=2\sqrt{1-x^2}$, тогда

$$y'=\cancel{2}\cdot\frac{1}{\cancel{2}\sqrt{1-x^2}}\cdot(-2x)=-\frac{2x}{\sqrt{1-x^2}}$$

$$1+(y')^2=1+\left(\frac{-2x}{\sqrt{1-x^2}}\right)^2=1+\frac{4x^2}{1-x^2}=\frac{1-x^2+4x^2}{1-x^2}=$$

$$=\frac{1+3x^2}{1-x^2}$$

$$Q_{ox}=2\cdot 2\pi\int_0^1 \underbrace{2\sqrt{1-x^2}}_{y(x)}\cdot\underbrace{\sqrt{\frac{1+3x^2}{1-x^2}}}_{\sqrt{1+(y'_x)^2}}\,dx=$$

$$=8\pi\int_0^1\sqrt{1+3x^2}\,dx =$$

$$\int_0^1 \sqrt{1+3x^2}\,dx = x\sqrt{1+3x^2}\Big|_0^1 - \int_0^1 x\cdot\frac{1\cdot 6x}{2\sqrt{1+3x^2}}\,dx =$$

$\| \; J$

интегрирование по частям

$$= \left(1\cdot\sqrt{1+3\cdot1^2} - 0\cdot\sqrt{1+3\cdot0^2}\right) -$$

$$- \int_0^1 \frac{3x^2+1-1}{\sqrt{1+3x^2}}\,dx = 2 - \underbrace{\int_0^1 \sqrt{1+3x^2}\,dx}_{J} +$$

$$+ \int_0^1 \frac{dx}{\sqrt{1+3x^2}}$$

$$2J = 2 + \frac{1}{\sqrt{3}}\int_0^1 \frac{d(\sqrt{3}x)}{\sqrt{1+(\sqrt{3}x)^2}} = 2 + \frac{1}{\sqrt{3}}\ln\left|\sqrt{3}x+\sqrt{1+3x^2}\right|\Big|_0^1$$

$$J = 1 + \frac{1}{2\sqrt{3}}\left(\ln|\sqrt{3}+2| - \ln 1\right) =$$

$$= 1 + \frac{1}{2\sqrt{3}}\ln|2+\sqrt{3}|$$

$$= 8\pi\left(1 + \frac{1}{2\sqrt{3}}\ln|2+\sqrt{3}|\right)$$

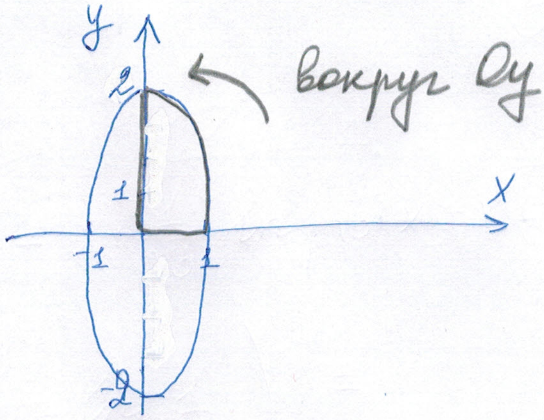
Ответ.

б) Q_{oy} – ?

$$4x^2+y^2=4$$
$$4x^2=4-y^2$$
$$x^2=1-\frac{1}{4}y^2$$
$$x=\pm\sqrt{1-\frac{y^2}{4}}$$

Пусть $x=\sqrt{1-\frac{y^2}{4}}$, тогда $x'_y=\frac{1}{2\sqrt{1-\frac{y^2}{4}}}\cdot\left(-\frac{1}{4}\right)\cdot 2y=$

$$=\frac{-\frac{1}{4}y}{\sqrt{1-\frac{y^2}{4}}}$$



$y\in[-2;2]$

М/о вычислить половины площади поверхности, а затем умножить на два.

$y\in[0;2]$

$$1+(x'_y)^2=1+\frac{\frac{y^2}{16}}{1-\frac{y^2}{4}}=\frac{1-\frac{y^2}{4}+\frac{y^2}{16}}{1-\frac{y^2}{4}}=$$

$$=\frac{1-\frac{3y^2}{16}}{1-\frac{y^2}{4}}$$

$$Q_{oy}=2\cdot 2\pi\int_0^2 \underbrace{\sqrt{1-\frac{y^2}{4}}}_{|x|}\cdot\underbrace{\frac{\sqrt{1-\frac{3y^2}{16}}}{\sqrt{1-\frac{y^2}{4}}}}_{\sqrt{1+(x'_y)^2}}\,dy=$$

$$= 4\pi \int_0^2 \frac{1}{4}\sqrt{16-3y^2}\,dy = \pi \int_0^2 \sqrt{16-3y^2}\,dy \ (=)$$

$$J = \int_0^2 \sqrt{16-3y^2}\,dy = y\sqrt{16-3y^2}\Big|_0^2 - \int_0^2 y \cdot \frac{1\cdot(-6y)}{2\sqrt{16-3y^2}}\,dy =$$

$$= \left(2\sqrt{16-12} - 0\cdot\sqrt{16-3\cdot 0}\right) - \int_0^2 \frac{-3y^2+16-16}{\sqrt{16-3y^2}}\,dy =$$

$$= 4 - \underbrace{\int_0^2 \sqrt{16-3y^2}\,dy}_{J} + 16\int_0^2 \frac{dy}{\sqrt{16-3y^2}} \Rightarrow$$

$$2J = 4 + 16\int_0^2 \frac{dy}{\sqrt{16-3y^2}} = 4 + 16\cdot\frac{1}{\sqrt{3}}\int_0^2 \frac{d(\sqrt{3}y)}{\sqrt{4^2-(\sqrt{3}y)^2}} =$$

$$= 4 + \frac{16}{\sqrt{3}} \arcsin\frac{\sqrt{3}y}{4}\Big|_0^2 = 4 + \frac{16}{\sqrt{3}}\left(\arcsin\frac{2\sqrt{3}}{4} - \right.$$

$$\left. - \arcsin 0\right) = 4 + \frac{16}{\sqrt{3}}\arcsin\frac{\sqrt{3}}{2} =$$

$$= 4 + \frac{16}{\sqrt{3}}\cdot\frac{\pi}{3}$$

$$J = 2 + \frac{8}{3\sqrt{3}}\pi$$

$$(=)\ 2\pi + \frac{8\pi^2}{3\sqrt{3}}$$ Ответ

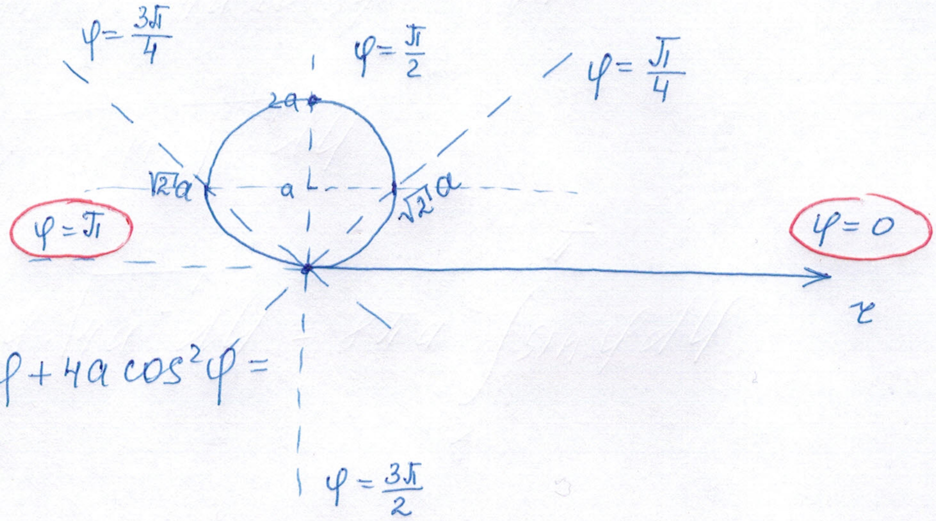
6.529 $r = 2a\sin\varphi$, $a > 0$

Q_{oy} – ?

О.Д.З. $r \geq 0$, $\sin\varphi \geq 0$

$\varphi \in [0 + 2\pi k, \pi + 2\pi k]$, $k \in \mathbb{Z}$

φ	0	$\frac{\pi}{2}$	π	$\frac{\pi}{4}$
r	0	$2a$	0	$\sqrt{2}a$



$r'_\varphi = 2a\cos\varphi$

$r^2 + (r'_\varphi)^2 = 4a^2\sin^2\varphi + 4a\cos^2\varphi =$
$= 4a^2$

$$Q_{oy} = 2\pi \int_0^\pi \underbrace{2a\sin\varphi}_{r} \cdot \sin\varphi \cdot \underbrace{\sqrt{(2a\sin\varphi)^2 + (2a\cos\varphi)^2}}_{\sqrt{r^2 + (r'_\varphi)^2}}\, d\varphi =$$

$$= 2\pi \int_0^\pi 2a\sin^2\varphi \cdot 2a\, d\varphi = 8\pi a^2 \int_0^\pi \sin^2\varphi\, d\varphi =$$

$$= 8\pi a^2 \int_0^\pi \frac{1 - \cos 2\varphi}{2}\, d\varphi = 4\pi a^2 \left[\int_0^\pi d\varphi - \int_0^\pi \cos 2\varphi\, d\varphi \right] =$$

$$= 4\pi a^2 \left[\varphi \Big|_0^\pi - \frac{1}{2} \int_0^\pi \cos 2\varphi\, d(2\varphi) \right] =$$

$$= 4\pi a^2 \left[\pi - \frac{1}{2} \sin 2\varphi \Big|_0^\pi \right] =$$

$$= 4\pi a^2 \left[\pi - \frac{1}{2} (\underset{0}{\underset{\parallel}{\sin 2\pi}} - \underset{0}{\underset{\parallel}{\sin 0}}) \right] = 4\pi^2 a^2$$

Ответ.

[ВОС] стр.

143 $\begin{cases} x=e^t\sin t, \\ y=e^t\cos t \end{cases}$ $0\le t\le \frac{\pi}{2}$

Q_{ox} – ?

$$\begin{cases} x'_t=e^t\sin t+e^t\cos t, \\ y'_t=e^t\cos t-e^t\sin t \end{cases}$$

$$(x'_t)^2+(y'_t)^2=(e^t(\sin t+\cos t))^2+(e^t(\cos t-\sin t))^2=$$

$$=e^{2t}(\sin^2 t+\cancel{2\sin t\cos t}+\cos^2 t)+$$

$$+e^{2t}(\cos^2 t-\cancel{2\sin t\cos t}+\sin^2 t)=$$

$$=e^{2t}\cdot 2(\sin^2 t+\cos^2 t)=2e^{2t}$$

$$\sqrt{(x'_t)^2+(y'_t)^2}=\sqrt{2}\,e^t$$

$$Q_{ox}=2\pi\int_0^{\frac{\pi}{2}}e^t\cos t\cdot\sqrt{2}\,e^t dt=2\sqrt{2}\int_0^{\frac{\pi}{2}}e^{2t}\cos t\,dt \;=$$

$$\int_0^{\frac{\pi}{2}}e^{2t}\cos t\,dt=\int_0^{\frac{\pi}{2}}e^{2t}d(\sin t)=e^{2t}\sin t\Big|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\sin t\,d(e^{2t})=$$

$$=e^{2\cdot\frac{\pi}{2}}\underbrace{\sin\frac{\pi}{2}}_{1}-e^{2\cdot 0}\underbrace{\sin 0}_{0}-2\int_0^{\frac{\pi}{2}}e^{2t}\sin t\,dt=$$

$$=e^{\pi}+2\int_0^{\frac{\pi}{2}}e^{2t}d(\cos t)=e^{\pi}+2\Big(e^{2t}\cos t\Big|_0^{\frac{\pi}{2}}-$$

$$-\int_0^{\frac{\pi}{2}}\cos t\,d(e^{2t})\Big)=e^{\pi}+2\Big(e^{2\cdot\frac{\pi}{2}}\underbrace{\cos\frac{\pi}{2}}_{0}-e^{2\cdot 0}\underbrace{\cos 0}_{1}\Big)-$$

$$-2\int_0^{\frac{\pi}{2}} 2e^{2t}\cos t\,dt = e^{\pi}-2-4\int_0^{\pi/2} e^{2t}\cos t\,dt$$

$$\int_0^{\pi/2} e^{2t}\cos t\,dt = I$$

$$I + 4I = e^{\pi} - 2$$

$$5I = e^{\pi} - 2$$

$$I = \frac{1}{5}e^{\pi} - \frac{2}{5}$$

$$= 2\sqrt{2}\left(\frac{1}{5}e^{\pi} - \frac{2}{5}\right)$$ Ответ