Вариант № 5

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Задача № 1

$(y')^3 \cdot y'' = 2y(y^2+2)^4$; $y(0)=0$; $y'(0)=2$

замена: $\begin{cases} y' = p(y) \\ y'' = p \cdot p' \end{cases}$

$p^3 \cdot p \cdot p' = 2y(y^2+2)^4$

$p^4 \, dp = 2y(y^2+2)^4 \, dy$

$\int p^4 \, dp = \int 2y(y^2+2)^4 \, dy$

$\frac{p^5}{5} = \int (y^2+2)^4 \, d(y^2+2)$

$\frac{p^5}{5} = \frac{(y^2+2)^5}{5} + C$

$p^5 = (y^2+2)^5 + C$

$32 = (0+2)^5 + C$

$C = 0$

$p = y^2+2$

$y' = y^2+2$

$\frac{dy}{dx} = y^2+2$

$\frac{dy}{y^2+2} = dx$

$\frac{1}{\sqrt{2}} \operatorname{arctg} \frac{y}{\sqrt{2}} = x + C_1$

$C_1 = 0$

$$x = \frac{1}{\sqrt{2}} \operatorname{arctg} \frac{y}{\sqrt{2}}$$

$$x\sqrt{2} = \operatorname{arctg} \frac{y}{\sqrt{2}}$$

$$\frac{y}{\sqrt{2}} = \operatorname{tg}(x\sqrt{2})$$

Odlem: $y = \sqrt{2} \cdot \operatorname{tg}(x\sqrt{2})$

Задача №2

$y''x^2\ln x - y'x + \ln^2 x = 0$

замена: $\begin{cases} y' = p(x) \\ y'' = p' \end{cases}$

$p'x^2\ln x - px + \ln^2 x = 0$

$p' - \frac{p}{x\ln x} + \frac{\ln x}{x^2} = 0$

Замена: $p = u(x)v(x)$

$p' = u'v + uv'$

$u'v + uv' - \frac{uv}{x\ln x} + \frac{\ln x}{x^2} = 0$

$v\left(u' - \frac{u}{x\ln x}\right) = -uv' - \frac{\ln x}{x^2}$

Пусть $\left(u' - \frac{u}{x\ln x}\right) = 0$

$u' = \frac{u}{x\ln x}$

$\frac{du}{dx} = \frac{u}{x\ln x}$

$\frac{du}{u} = \frac{dx}{x\ln x}$

$\int \frac{du}{u} = \int \frac{dx}{x\ln x}$

$$\ln u = \ln \ln x$$

$$u_{\text{частн}} = \ln x$$

$$u v' + \frac{\ln x}{x^2} = 0$$

$$v' \ln x + \frac{\ln x}{x^2} = 0$$

$$\ln x \frac{dv}{dx} + \frac{\ln x}{x^2} = 0$$

$$\frac{dv}{dx} + \frac{1}{x^2} = 0$$

$$dv = -\frac{dx}{x^2}$$

$$v = \frac{1}{x} + C_1$$

$$P = u \cdot v = \frac{\ln x}{x} + \ln x\, C_1$$

$$y' = \frac{\ln x}{x} + \ln x\, C_1$$

$$y = \int \left(\frac{\ln x}{x} + \ln x\, C_1\right) dx$$

$$y = \int \frac{\ln x}{x} dx + \int \ln x\, C_1\, dx$$

$$y = \frac{\ln^2 x}{2} + C_1\left(x \ln x - \int x\, d\ln x\right)$$

Ответ: $y = \frac{\ln^2 x}{2} + C_1 x \ln x - C_1 x + C$

Задача №3

$$y^{IV} - y'' = \underbrace{e^{-x}\sin x}_{f_1(x)} + \underbrace{x^2 e^x}_{f_2 x} + \underbrace{x\cos 2x - \sin 2x}_{f_3(x)} + \underbrace{1-x}_{f_4(x)}$$

1. ЛОДУ: $y^{IV} - y'' = 0$

Хар-ое ур-е: $k^4 - k^2 = 0$

$k^2(k-1)(k+1) = 0$

$$\begin{cases} k_1 = -1 \\ k_2 = k_3 = 0 \\ k_4 = 1 \end{cases}$$

ФСР: $\{y_1 = e^{-x};\ y_2 = e^{0x};\ y_3 = e^{0x}\cdot x;\ y_4 = e^{x}\}$

$y_{oo} = C_1 y_1 + C_2 y_2 + C_3 y_3 + C_4 y_4$

общее реш-е ЛОДУ:

$y_{oo} = C_1\cdot e^{-x} + C_2\cdot e^{0x} + C_3\cdot e^{0x}\cdot x + C_4\cdot e^{x}$

2. Распишем правую часть ЛНДУ:

$f_1(x) = e^{-x}\cdot\sin x = e^{-1\cdot x}(1\cdot\sin x + 0\cdot\cos x) \Rightarrow \begin{cases} S = 0 \\ \alpha \pm \beta i = -1 \pm i \\ r = 0 \end{cases}$

$y_{ч1} = e^{-x}(A_1\cdot\sin x + B_1\cos x)\cdot x^0$

$+1-x$
$f_4(x)$

$$f_2(x) = x^2 e^x \Rightarrow \begin{cases} n=2 \\ \alpha=1 \\ r=1 \end{cases} \Rightarrow y_{чн2} = e^x(A_2x^2+B_2x+C_2)\cdot x$$

$$f_3(x) = x\cdot\cos 2x - \sin 2x \Rightarrow \begin{cases} s=1 \\ \alpha\pm\beta i = 0\pm i \\ r=0 \end{cases} \Rightarrow$$

$$\Rightarrow y_{чн3} = e^{0x}((A_3x+B_3)\cos 2x + (A_4x+B_4)\sin 2x)\cdot x^0$$

$$f_4(x) = 1-x \Rightarrow \begin{cases} n=1 \\ \alpha=0 \\ r=2 \end{cases} \Rightarrow$$

$$\Rightarrow y_{чн4} = e^{0x}(A_5x+B_5)\cdot x^2$$

Ответ:

$$y_{он} = C_1e^{-x} + C_2\cdot e^{0x} + C_3\cdot e^{0x}\cdot x + C_4e^x +$$
$$+ e^{-x}(\sin x\cdot A_1 + B_1\cdot\cos x)x^0 + e^x(A_2x^2+B_2x+C_2)x +$$
$$+ e^{0x}((A_3x+B_3)\cos 2x + (A_4x+B_4)\sin 2x)\cdot x^0 +$$
$$+ e^{0x}(A_5x+B_5)\cdot x^2$$

Задача №4

$$\underbrace{y''+8y'+16y=4e^{-4x}}_{f_1(x)}\underbrace{-16x+8}_{f_2(x)} \qquad y(0)=-1, \quad y'(0)=8$$

1. ЛОДУ: $y''+8y'+16y=0$

Хар. ур-е: $k^2+8k+16=0$

$(k+4)^2=0 \Rightarrow k_1=-4$, $k_2=-4$

ФСР: $\{y_1=e^{-4x} \quad y_2=e^{-4x}\cdot x\}$

$y_{oo}=C_1e^{-4x}+C_2e^{-4x}\cdot x$

2. Рассмотрим по отдельности правую часть ЛНДУ:

$$f_1(x)=4\cdot x^0\cdot e^{-4x} \Rightarrow \begin{cases} n=0 \\ \alpha=-4 \\ r=2 \end{cases} \Rightarrow$$

$\Rightarrow y_{чн1}=A\cdot e^{-4x}\cdot x^2$

$$\begin{cases} y=A\cdot e^{-4x}\cdot x^2 \\ y'=-4Ae^{-4x}\cdot x^2+2Ax\cdot e^{-4x}=2Ae^{-4x}(x-2x^2) \\ y''=-8Ae^{-4x}(x-2x^2)+2Ae^{-4x}(1-4x)= \end{cases}$$

$=2Ae^{-4x}(8x^2-8x+1)$

$$y'' + 8y' + 16y = 4e^{-4x}$$

$$2Ae^{-4x}(8x^2 - 8x + 1) + 16Ae^{-4x}(x - 2x^2) +$$
$$+ 16 \cdot A \cdot e^{-4x} \cdot x^2 = 4e^{-4x}$$

$$2Ae^{-4x}(8x^2 - 8x + 1 + 8x - 16x^2 + 8x^2) = 4e^{-4x}$$

$$2Ae^{-4x} = 4e^{-4x} \Rightarrow A = 2$$

$$y_{чн1} = 2e^{-4x} \cdot x^2$$

$$f_2(x) = (-16x + 8) \cdot e^{0x} \Rightarrow \begin{cases} n = 1 \\ \alpha = 0 \\ r = 0 \end{cases}$$

$$y_{чн2} = e^{0x}(Bx + C) \cdot x^0 \Rightarrow y_{чн2} = Bx + C$$

$$\begin{cases} y = Bx + C \\ y' = B \\ y'' = 0 \end{cases}$$

$$y'' + 8y' + 16y = -16x + 8$$

$$0 + 8B + 16Bx + 16C = -16x + 8$$

$$x^1: \ 16B = -16 \Rightarrow B = -1$$

$$x^0: \ 8B + 16C = 8 \Rightarrow C = 1$$

$$y_{чн2} = -x + 1$$

3. $y_{он} = y_{оо} + y_{чн1} + y_{чн2}$

$$\begin{cases} y = C_1 \cdot e^{-4x} + C_2 e^{-4x} \cdot x + 2e^{-4x} \cdot x^2 - x + 1 \\ y' = -4C_1 e^{-4x} - 4C_2 e^{-4x} \cdot x + C_2 e^{-4x} + 2 \cdot (-4) e^{-4x} x^2 + 2 \cdot 2 \cdot e^{-4x} \cdot x - 1 \end{cases}$$

$$\begin{cases} -1 = C_1 + 0 + 0 - 0 + 1 \\ 8 = -4C_1 + C_2 - 1 \end{cases} \qquad \begin{cases} C_1 = -2 \\ C_2 = 1 \end{cases}$$

Ответ: $y = -2e^{-4x} + e^{-4x} \cdot x + 2e^{-4x} \cdot x^2 - x + 1$

Задача №5

$y'' - (1+2\,tg\,x)y' + (tg\,x - 1)y = 2e^{2x} \sec x$

$y'' - (1+2\,tg\,x)y' + (tg\,x - 1)y = \frac{2e^{2x}}{\cos x}$

$\begin{cases} P_1(x) = -1 - 2\,tg\,x \\ f(x) = \frac{2e^{2x}}{\cos x} \end{cases}$

$y_2 = \frac{1}{\cos x} \cdot \int \frac{e^{\int (1+2\,tg\,x)dx}}{(1/\cos x)^2}\, dx$

$\int (1+2\,tg\,x)\,dx = \int dx + 2\int \frac{\sin x}{\cos x} dx =$

$= \int dx + 2\int \frac{-d(\cos x)}{\cos x} = x - 2\ln(\cos x) =$

$= \ln e^x - \ln \cos^2 x = \ln \frac{e^x}{\cos^2 x}$

$y_2 = \frac{1}{\cos x} \cdot \int \frac{e^x}{\cos^2 x} \cdot \cos^2 x\, dx \Rightarrow y_2 = \frac{e^x}{\cos x} + C$

ФСР ЛОДУ: $\{ y_1 = \frac{1}{\cos x};\ y_2 = \frac{e^x}{\cos x} \}$

$y_{oo} = C_1 \cdot \frac{1}{\cos x} + C_2 \cdot \frac{e^x}{\cos x}$

2. Пусть $C_1 = C_1(x)$, $C_2 = C_2(x)$

$$y_{он} = C_1(x)\cdot\frac{1}{\cos x} + C_2(x)\cdot\frac{e^x}{\cos x}$$

$$\begin{cases} C_1'(x)\,y_1 + C_2'(x)\,y_2 = 0 \\ C_1'(x)\,y_1' + C_2'(x)\,y_2' = f(x) \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} C_1'(x)\cdot\frac{1}{\cos x} + C_2'(x)\cdot\frac{e^x}{\cos x} = 0 \\ C_1'(x)\cdot\left(\frac{1}{\cos x}\right)' + C_2'(x)\cdot\left(\frac{e^x}{\cos x}\right)' = \frac{2e^{2x}}{\cos x} \end{cases}$$

По правилу Крамера:

$$\Delta = \begin{vmatrix} \frac{1}{\cos x} & \frac{e^x}{\cos x} \\ \frac{\sin x}{\cos^2 x} & \frac{e^x\cos x + \sin x\, e^x}{\cos^2 x} \end{vmatrix} =$$

$$= \frac{e^x\cos x + \sin x\cdot e^x - e^x\sin x}{\cos^3 x} = \frac{e^x}{\cos^2 x}$$

$$\Delta_1 = \begin{vmatrix} 0 & \frac{e^x}{\cos x} \\ \frac{2e^{2x}}{\cos x} & \frac{e^x\cos x + \sin x\, e^x}{\cos^2 x} \end{vmatrix} = -\frac{e^x\cdot 2e^{2x}}{\cos^2 x}$$

$$\Delta_2 = \begin{vmatrix} \frac{1}{\cos x} & 0 \\ \frac{\sin x}{\cos^2 x} & \frac{2e^{2x}}{\cos x} \end{vmatrix} = \frac{2e^{2x}}{\cos^2 x}$$

$$C_1'(x) = -\frac{e^x \cdot 2e^{2x} \cdot \cos^2 x}{\cos^2 x \cdot e^x} = -2e^{2x}$$

$$C_2'(x) = \frac{2e^{2x}}{\cos^2 x} \cdot \frac{\cos^2 x}{e^x} = 2e^x$$

$$C_1(x) = -2\int e^{2x}\,dx = -\int e^{2x}\,d(2x) = -e^{2x} + C_1$$

$$C_2(x) = 2\int e^x = 2e^x + C_2$$

Ответ: $y_{он} = (-e^{2x} + C_1) \cdot \frac{1}{\cos x} + (2e^x + C_2) \cdot \frac{e^x}{\cos x}$