Вариант 8

Кудинов Леонид

№1

$$(y^2+xy^2)y'+x^2-yx^2=0$$

$$y^2(1+x)y'+x^2(1-y)=0$$

$$y^2(1+x)dy=x^2(y-1)dx$$

$$\frac{y^2dy}{(y-1)}=\frac{x^2dx}{(1+x)}$$ — ур-е с разделяющимися переменными

$$\int\frac{y^2dy}{y-1}=\int\frac{x^2dx}{x+1}$$

$$\int\frac{y^2dy}{y-1}=\int\left(y+\frac{1}{y-1}+1\right)dy=\int ydy+\int\frac{d(y-1)}{y-1}+\int 1dy$$

$$=\frac{y^2}{2}+\ln|y-1|+y$$

$$\int\frac{x^2dx}{x+1}=\int\left(x+\frac{1}{x+1}-1\right)=\int xdx+\int\frac{d(x+1)}{x+1}-\int 1dx=$$

$$=\frac{x^2}{2}+\ln|x+1|-x+C$$

$\ln|y-1| + \frac{y^2}{2} + y = \ln|x+1| + \frac{x^2}{2} - x + C$

Потерянные решения $x=-1$, $y=1$

Ответ: $\ln|y-1| + \frac{y^2}{2} + y = \ln|x+1| + \frac{x^2}{2} - x + C$

$x = -1$, $y = 1$

№2

$(xy + x^2y^3)dy = dx$

$xy + x^2y^3 = x'_y$

$x'_y - xy = x^2y^3$ – ур-е Бернулли

$x(y) = u(y)v(y)$

$x'(y) = u'_y(y)v(y) + u(y)v'_y(y)$

$u'v + uv' - uvy = x^2y^3$

$v(u' - uy) = x^3y^3 - uv'$

$u' - uy = 0$

$\frac{du}{dy} = uy$

$\int \frac{du}{u} = \int y\,dy$

$\ln|u| = \frac{y^2}{2} + C$

$|u| = e^{\frac{y^2}{2}} \cdot C_1, \quad C_1 = e^C$

$u = e^{\frac{y^2}{2}} \cdot C_2, \quad C_2 = \pm C_1$

$u = e^{\frac{y^2}{2}}$

$e^{\frac{y^2}{2}} v' = \cancel{x}^{2} v^{-\cancel{x}2} y^3$

$v' = \frac{u^{\cancel{2}}}{u} v^{-\cancel{2}} y^3$

$v' = u v^2 y^3$

$\int \frac{dv}{v^{-2}} = \int e^{\frac{y^2}{2}} y^3 dy$

$-\frac{1}{v} = (y^2 - 2) e^{\frac{y^2}{2}} + C$

$v = \frac{1}{(2 - y^2) e^{\frac{y^2}{2}} - C}$

$\int e^{\frac{y^2}{2}} y^3 dy = e^{\frac{y^2}{2}} y^4 - \int y\, d(e^{\frac{y^2}{2}} y^3) =$

$\left| d(e^{\frac{y^2}{2}} y^3) = y e^{\frac{y^2}{2}} \cdot 3y^2 d(y) \right| = e^{\frac{y^2}{2}} y^4 - 3\int e^{\frac{y^2}{2}} y^4 dy =$

$= \left| \begin{matrix} t = y^2 \\ dy = \frac{1}{2y} dt \end{matrix} \right| = \frac{1}{2}\int t e^{\frac{t}{2}} dt = \left| \begin{matrix} z = -\frac{u}{2} \\ dt = -2dz \end{matrix} \right|$

$$= 4\int z e^{-z}\,dz = -4ze^{-z} - 4\int -e^{-z}\,dz =$$

$$= -\cancel{4}ze^{-z} - e^{-z}\cancel{4} = x^2 e^{\frac{x^2}{2}} - 2e^{\frac{x^2}{2}} = (x^2-2)e^{\frac{x^2}{2}}$$

Отв $x(y) = \dfrac{e^{\frac{y^2}{2}}}{(2-y^2)e^{\frac{y^2}{2}} + C}$

Потерянные решения: $x(y) = 0$

Ответ: $x(y) = \dfrac{e^{\frac{y^2}{2}}}{(2-y^2)e^{\frac{y^2}{2}} - C}$; $x(y) = 0$

№3

$$\frac{dx}{y+x} = \frac{dy}{y-x}, \quad y\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$$

$y' = \dfrac{y-x}{y+x}$ – однородное ур-е

$t = \dfrac{y}{x}$; $y = tx$; $y' = t'x + t$

$$t'x + t = \frac{t-1}{t+1}$$

$$\frac{dt}{dx}x = \frac{t-1}{t+1} - t$$

$$\int \frac{dt}{\frac{t-1}{t+1} - t} = \int \frac{dx}{x}$$

$$\int -\frac{t+1}{1+t^2} dt = \int \frac{dx}{x}$$

$$-\int \frac{t+1}{t^2+1} dt = -\int \frac{t}{t^2+1} dt - \int \frac{1}{t^2+1} dt = -\frac{1}{2} \int \frac{d(t^2)}{t^2+1} =$$

$$\int \frac{1}{t^2+1} d(t) = -\frac{1}{2} \frac{\ln |t^2+1|}{1} - \operatorname{arctg} t + C$$

$$\ln(x) = -\frac{1}{2} \ln|t^2+1| - \operatorname{arctg} t + C$$

$$\ln(x) = -\frac{1}{2} \ln \left|\frac{y^2}{x^2}+1\right| - \operatorname{arctg} \frac{y}{x} + C$$

$$\frac{1}{2} \ln(x^2+y^2) = -\operatorname{arctg}\left(\frac{y}{x}\right) + C$$

$$\frac{1}{2} \ln\left(\frac{1}{2}+\frac{1}{2}\right) = -\operatorname{arctg} 1 + C$$

$$\frac{1}{2} \ln 1 = -\operatorname{arctg} 1 + C$$

$$C = \operatorname{arctg} 1 \Rightarrow C = \frac{\pi}{4}$$

Ответ: $\frac{1}{2}\ln(x^2+y^2) = -\operatorname{arctg}\left(\frac{y}{x}\right) + C$

частное решение:

$$\frac{1}{2}\ln(x^2+y^2) = -\operatorname{arctg}\left(\frac{y}{x}\right) + \frac{\pi}{4}$$

№4

$$xy' = 2y + 2(\ln^2 x - \ln x), \quad y(1) = ?$$

$$y' - \frac{2y}{x} = \frac{2(\ln^2 x - \ln x)}{x}$$ – НОЛУ

$$y' - \frac{2y}{x} = 0$$

$$\frac{dy}{dx} = \frac{2y}{x}$$

$$\int\frac{dy}{y} = 2\int\frac{dx}{x}$$

$$\ln|y| = 2\ln|x| + C$$

$$|y| = |x|^2 C_1, \quad C_1 = e^C$$

$$y = C_2 x^2, \quad C_2 = \pm C_1$$

$$y' = C_2' x^2 + 2x C_2, \quad \text{где } C_2 = C(x)$$

$$C_2' x^2 + \cancel{2x C_2} - \cancel{\frac{2C_2 x^2}{x}} = \frac{2(\ln^2 x - \ln x)}{x}$$

$$\frac{dC_2}{dx} x^2 = \frac{2(\ln^2 x - \ln x)}{x}$$

$$\int dC_2 = \int \frac{2(\ln^2 x - \ln x)}{x^3} dx$$

$$C_2 = 2\int \frac{\ln^2 x - \ln x}{x^3} = 2\int \frac{\ln x(\ln x - 1)}{x^3} dx =$$

$$= 2\int \frac{(\ln x - 1)}{x^4} d(\ln x) = 2\int \frac{\ln x}{x^4} d(\ln x) - 2\int \frac{d(\ln x)}{x^4} =$$

$$= 2\int \frac{\ln x}{e^{\ln x}} d(\ln x) - 2\int \frac{d(\ln x)}{e^{4\ln x}} = 2\frac{\ln^2 x}{x^4} - 2\int \ln(x) d\left(\frac{\ln x}{x^4}\right)$$

$$-2\int \cancel{\frac{d(\ln x)}{x^4}}$$

$$2\int \frac{\ln^2 x - \ln x}{x^3}dx = 2\int \frac{\ln^2 x - \ln x}{e^{3\ln x}}dx =$$

$$\left| \begin{array}{l} t = \ln x \\ dx = x\,dt \end{array} \right| = 2\int (t^2 - t)e^{-2t}dt =$$

$$-\frac{2(t^2 - t)e^{-2t}}{2} + \cancel{2}\int \frac{(2t-1)e^{-2t}}{\cancel{2}}dt = I$$

$$I = \int (2t-1)\cdot e^{-2t}dt = -\frac{(2t-1)e^{-2t}}{2} - \int -e^{-2t}dt$$

$$= -\frac{(2t-1)e^{-2t}}{2} - \frac{e^{-2t}}{2}$$

$$-\frac{\cancel{2}(t^2 - t)e^{-2t}}{\cancel{2}} - \frac{(2t-1)e^{-2t}}{2} - \frac{e^{-2t}}{2}$$

Подставим $t = \ln x$

$$-\frac{(\ln^2 x - \ln)}{x^2} - \frac{(2\ln x - 1)}{2x^2} - \frac{1}{2x^2} =$$

$$= -\frac{\ln^2 x}{x^2} + C_3$$

$$C_2(x) = \frac{-\ln^2 x}{x^2} + C_3$$

$$y = -\ln^2 x + C_3 x^2$$

При $y(1) = 2$

$$2 = -\ln^2 1 + C_3$$

$$C_3 = 2$$

Ответ: $y = -\ln^2 x + C_3 x^2$

частное реш.

$$y = -\ln^2 x + 2x^2$$