С.з №4. Тема: Производная сложной ф-ции. Производная неявно заданной ф-ции. Производная по направлению. Градиент.

Производная сложной функции:

$z = f(x, y)$ - диф. ф-ция 2-х переменных x и y

$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}, \Rightarrow \quad z = f(\varphi(t), \psi(t)) \text{ - сложная функция}$$

ф-ция одной переменной

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}} \quad (1)$$

$z = f(x, y)$ - диф. ф-ция 2-х переменных x и y

$$y = y(x) \quad \Rightarrow \quad z = f(x, y(x)) \text{ - сложная функция}$$

ф-ция одной переменной

$$\boxed{\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx}} \quad (2)$$

$z = f(u, v)$ - диф-ма ф-ция 2-х переменных u и v, $\begin{cases} u = \varphi(x, y) \\ v = \psi(x, y) \end{cases}$, $\Longrightarrow$

$$\Rightarrow \boxed{z = f(\varphi(x,y), \psi(x,y))}$$ – сложная ф-ция

ф-ция 2-х переменных x и y

$$\begin{cases} \dfrac{\partial z}{\partial x} = \dfrac{\partial f}{\partial u}\cdot\dfrac{\partial u}{\partial x} + \dfrac{\partial f}{\partial v}\cdot\dfrac{\partial v}{\partial x}, \\ \dfrac{\partial z}{\partial y} = \dfrac{\partial f}{\partial u}\cdot\dfrac{\partial u}{\partial y} + \dfrac{\partial f}{\partial v}\cdot\dfrac{\partial v}{\partial y} \end{cases} \quad (3)$$

<u>Задачи:</u> 7.114, 7.119, 7.122, 7.129, 7.135 [Е.Д]

д/з 7.116, 7.118, 7.123, 7.130, 7.136

7.114 $z = e^{2x-3y}$, $x = \operatorname{tg} t$, $y = t^2 - t$

$$\left.\begin{array}{l} z = f(x,y) \\ x = x(t) \\ y = y(t) \end{array}\right\} \Rightarrow f(x(t), y(t)) = z$$

сложная ф-ция одной переменной t

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}} \quad - (1)$$

$$\frac{dz}{dt} = e^{2x-3y} \cdot 2 \cdot \frac{1}{\cos^2 t} + e^{2x-3y} \cdot (-3) \cdot (2t-1) =$$

$$= e^{2x-3y} \left(\frac{2}{\cos^2 t} - 6t + 3 \right)$$

$$\frac{\partial z}{\partial x} = 2 \cdot e^{2x-3y} \qquad \frac{dx}{dt} = \frac{1}{\cos^2 t}$$

$$\frac{\partial z}{\partial y} = -3 \cdot e^{2x-3} \qquad \frac{dy}{dt} = 2t - 1$$

7.119 $\quad z = \operatorname{arctg} \frac{x-1}{y}$

$$y = e^{(x+1)^2}$$

$$\left.\begin{array}{l} z = f(x,y) \\ y = y(x) \end{array}\right\} \Rightarrow z = f(x, y(x))$$ – сложная ф-ция одной переменной

$$\boxed{\frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}}$$

$$\frac{\partial f}{\partial x} = \frac{1}{1 + \left(\frac{x-1}{y}\right)^2} \cdot \frac{1}{y} = \frac{y^{\cancel{2}}}{y^2 + (x-1)^2} \cdot \frac{1}{\cancel{y}} =$$

$$= \frac{y}{y^2 + (x-1)^2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{x-1}{y}\right)^2} \cdot (x-1) \cdot \left(-\frac{1}{y^2}\right) =$$

$$= \frac{1-x}{y^2+(x-1)^2} \cdot \frac{y^2}{y^2} = \frac{1-x}{y^2+(x-1)^2}$$

$$\frac{dy}{dx} = 2(x+1)e^{(x+1)^2}$$

$$\frac{dz}{dx} = \frac{y}{y^2+(x-1)^2} + \frac{1-x}{y^2+(x-1)^2} \cdot 2(x+1)e^{(x+1)^2} =$$

$$= \frac{1}{y^2+(x-1)^2}\left(y + 2(1-x)(x+1)e^{(x+1)^2}\right) =$$

$$= \frac{1}{y^2+(x-1)^2}\left(y - 2(x^2-1)e^{(x+1)^2}\right)$$

7.122 $z = f(u, v)$

$$u = \frac{2y}{x+y}$$

$$v = x^2 - 3y$$

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Сложная ф-ция: $z = f(u(x,y), v(x,y))$

ф-ция 2-х переменных x и y.

$$\begin{cases} \frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}, \\ \frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} \end{cases}$$

$$u'_x = -\frac{2y}{(x+y)^2}, \quad v'_x = 2x$$

$$u'_y = 2\,\frac{x+y-2y}{(x+y)^2} = 2\cdot\frac{x-y}{(x+y)^2}, \quad v'_y = -3$$

$$\begin{cases} \frac{\partial z}{\partial x} = f'_u\cdot\left(-\frac{2y}{(x+y)^2}\right) + f'_v\cdot 2x \\ \frac{\partial z}{\partial y} = f'_u\cdot 2\cdot\frac{x-y}{(x+y)^2} + f'_v\cdot(-3) \end{cases}$$

$$\begin{cases} \frac{\partial z}{\partial x} = -\frac{2y}{(x+y)^2} f'_u + 2x f'_v \\ \frac{\partial z}{\partial y} = 2\,\frac{x-y}{(x+y)^2} f'_u - 3 f'_v \end{cases}$$

7.129 $z = x \cdot f\left(\underbrace{\frac{y}{x}}_{=t}\right) - x^2 - y^2$

Док-ть равенство: $xz'_x + yz'_y = z - x^2 - y^2$

$$z'_x = f\left(\frac{y}{x}\right) + x\cdot f'\left(\frac{y}{x}\right)\cdot \cancel{f}\left(\frac{y}{x}\right)'_x - 2x =$$

$$= f + xf'\cdot y\cdot\left(-\frac{1}{x^2}\right) - 2x =$$

$$= f - \frac{y}{x}f' - 2x$$

$$z'_y = x\cdot f'\left(\frac{y}{x}\right)\cdot\left(\frac{y}{x}\right)'_y - 2y =$$

$$= x\cdot\frac{1}{x}\cdot f' - 2y = f' - 2y$$

$$xf \cancel{- x\cdot\frac{y}{x}f'} - 2x\cdot x \cancel{+ yf'} - 2y\cdot y =$$

$$= z - x^2 - y^2$$

$$xf - 2x^2 - 2y^2 = z - x^2 - y^2 =$$

$$= xf - x^2 - y^2 - x^2 - y^2 =$$

$$= xf - 2x^2 - 2y^2$$

ч.т.д.

7.135 $u = x\varphi(\overbrace{x+y}^{=t}) + y\psi(\overbrace{x+y}^{=t})$

Док-ть рав-во: $u''_{xx} - 2u''_{xy} + u''_{yy} = 0$

$$u'_x = \varphi(x+y) + x\cdot\varphi'(x+y) + y\psi'(x+y)$$

$$u''_{xx} = \varphi'(x+y) + \varphi'(x+y) + x\cdot\varphi''(x+y) + y\psi''(x+y) =$$

$$= 2\varphi'(x+y) + x\varphi''(x+y) + y\psi''(x+y)$$

$$u'_y = x\varphi'(x+y) + \psi(x+y) + y\psi'(x+y)$$

$$u''_{yy} = x\varphi''(x+y) + \psi'(x+y) + \psi'(x+y) + y\psi''(x+y) =$$

$$= x\varphi''(x+y) + 2\psi'(x+y) + y\psi''(x+y)$$

$$u''_{xy} = \left(\varphi(x+y) + x\varphi'(x+y) + y\psi'(x+y)\right)'_y =$$

$$= \varphi' + x\varphi'' + \psi' + y\psi''$$

$$\underbrace{2\cancel{\varphi'} + \cancel{x\varphi''} + y\cancel{\psi''}}_{u''_{xx}} \underbrace{- 2\cancel{x\varphi''} - \cancel{2\varphi'} - 2\cancel{\psi'} - 2\cancel{y\psi''}}_{-2u''_{xy}} +$$

$$+ \underbrace{\cancel{x\varphi''} + 2\cancel{\psi'} + y\cancel{\psi''}}_{u''_{yy}} = 0$$

$0=0$

Производная неявно заданной функции

$f(x,y)=0$

$y=\varphi(x)$

$f(x,\varphi(x))=0$

$$\boxed{y'_x=-\frac{f'_x}{f'_y}}$$

$f'_y \neq 0$.

$F(x,y,z)=0$

$z=f(x,y)$

$$\boxed{z'_x=-\frac{F'_x(x,y,z)}{F'_z(x,y,z)}}$$

$$\boxed{z'_y=-\frac{F'_y(x,y,z)}{F'_z(x,y,z)}}$$

$F'_z(x,y,z)\neq 0$

Задачи: 7.141, 7.145, 7.149, 7.152

Д/З 7.140, 7.146, 7.150, 7.151.

7.141 $y\sin x-\cos(x-y)=0$

y'_x – ?

$f(x,y)=y\sin x-\cos(x-y)=0$

$$y'_x=-\frac{f'_x}{f'_y}=-\frac{y\cos x+\sin(x-y)}{\sin x+\cos(x-y)}$$

7.145 $\quad z^3 - 4xz + y^2 - 4 = 0.$ $\quad M(1, -2, 2)$

$F(x, y, z) = 0$

$f(x, y) = z$

$$\begin{cases} z'_x = -\dfrac{F'_x}{F'_z} = -\dfrac{-4z}{3z^2 - 4x}, \\ z'_y = -\dfrac{F'_y}{F'_z} = -\dfrac{2y}{3z^2 - 4x} \end{cases}$$

$$\begin{cases} F'_x = -4z, \\ F'_y = 2y, \\ F'_z = 3z^2 - 4x. \end{cases}$$

$$z'_x = \frac{4z}{3z^2 - 4x}\Bigg|_{M(1,-2,2)} = \frac{4 \cdot 2}{3 \cdot 4 - 4 \cdot 1} = \frac{8}{12 - 4} = \frac{8}{8} = 1$$

$$z'_y = -\frac{2y}{3z^2 - 4x}\Bigg|_{M(1,-2,2)} = \frac{-2 \cdot (-2)}{3 \cdot 4 - 4 \cdot 1} = \frac{4}{12 - 4} = \frac{4}{8} = \frac{1}{8}$$

7.149

$yz = \operatorname{arctg}(x z)$ $\qquad dz - ?$

$yz - \operatorname{arctg}(xz) = 0$

$F(x, y, z) = 0$

$$\boxed{F(x, y, z) = yz - \operatorname{arctg}(xz) = 0}$$

$z = f(x, y)$

$$\boxed{dz = z'_x\,dx + z'_y\,dy}$$

$$\begin{cases} z'_x = -\dfrac{F'_x}{F'_z}, \\ z'_y = -\dfrac{F'_y}{F'_z} \end{cases}$$

$$\begin{cases} F'_x = -\dfrac{1}{1+x^2z^2}\cdot z \\ F'_y = z \\ F'_z = y - \dfrac{1}{1+x^2z^2}\cdot x \end{cases}$$

$$dz = \left(\frac{x}{1+x^2z^2} - y\right)\left(-\frac{z}{1+x^2z^2}\,dx + z\,dy\right)$$

7.152

$$x+y+z=e^z \qquad z''_{xx}-?$$

$$x+y+z-e^z=0 \qquad z''_{xy}-?$$

$$\boxed{F(x,y,z)=x+y+z-e^z} \qquad z''_{yy}-?$$

$$z=f(x,y)$$

$$\begin{cases} F'_x=1, \\ F'_y=1, \\ F'_z=1-e^z \end{cases}$$

$$z'_x=-\frac{F'_x}{F'_z}=-\frac{1}{1-e^z}$$

$$z'_y=-\frac{F'_y}{F'_z}=-\frac{1}{1-e^z}$$

$$z''_{xx}=\left(\frac{1}{e^z-1}\right)'_x=-\frac{1\cdot e^z}{(e^z-1)^2}\cdot z'_x=-\frac{1\cdot e^z}{(e^z-1)^2}\cdot -\frac{1}{1-e^z}=$$

$$=-\frac{1\cdot e^z}{(e^z-1)^2}\cdot\frac{1}{(e^z-1)}=-\frac{1\cdot e^z}{(e^z-1)^3}$$

$$z''_{yy}=-\frac{e^z}{(e^z-1)^3}$$

$$z''_{xy}=-\frac{e^z}{(e^z-1)^3}$$

Производная по направлению. Градиент.

Опр. Пусть в некоторой области D задана функция $u = u(x, y, z)$ и точка $M(x, y, z)$. Проведём из точки M вектор $\vec{s}$, направляющие косинусы которого $\cos\alpha$, $\cos\beta$, $\cos\gamma$. На векторе $\vec{s}$ на расстоянии Δs отметим точку $M_1(x+\Delta x, y+\Delta y, z+\Delta z)$, т. е. $\Delta s = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$. Будем предполагать, что u и её первые частные производные непрерывны в D. Тогда производной по направлению $\vec{s}$ ф-ции $u = u(x, y, z)$ наз-ся

$$\boxed{\lim_{\Delta s \to 0} \frac{\Delta u}{\Delta s} = \frac{\partial u}{\partial \vec{s}}}$$

Правило:
$$\boxed{\left.\frac{\partial u}{\partial s}\right|_{M_0} = \left.\frac{\partial u}{\partial x}\right|_{M_0} \cdot \cos\alpha + \left.\frac{\partial u}{\partial y}\right|_{M_0} \cos\beta + \left.\frac{\partial u}{\partial z}\right|_{M_0} \cdot \cos\gamma}$$

$\cos\alpha$, $\cos\beta$, $\cos\gamma$ – направляющие косинусы $\vec{s}$

$$\vec{s} = \{ s_x, s_y, s_z \}$$

$$\cos\alpha = \frac{s_x}{|\vec{s}|}, \quad \cos\beta = \frac{s_y}{|\vec{s}|}, \quad \cos\gamma = \frac{s_z}{|\vec{s}|}$$

Градиент функции – это вектор, проекциями которого на координатные оси яв-ся значения частных производных этой функции в соответствующей точке: $u = u(x, y, z)$

$$\operatorname{grad} u = \nabla u = \left\{ \frac{\partial u}{\partial x}; \frac{\partial u}{\partial y}; \frac{\partial u}{\partial z} \right\} = \{ u'_x, u'_y, u'_z \} =$$

$$= \boxed{u'_x \vec{i} + u'_y \vec{j} + u'_z \cdot \vec{k} = \nabla u}$$

Задачи: [E.D] часть 2.

10.31
10.33
10.35
10.37
10.39.

Д/З
10.32
10.34
10.38
10.36.

10.31 $u = x^2 + \frac{1}{2} y^2$

$P_0 (2, -1)$

$P_1 (6, 2)$

$\frac{\partial u}{\partial \overrightarrow{P_0 P_1}}$ – ?

$$\vec{S}$$
$$\parallel$$

$\overrightarrow{P_0 P_1} = \{ 6-2; 2+1 \} = \{ 4, 3 \}$ – направляющий вектор

$|\vec{s}| = \sqrt{4^2 + 3^2} = 5$

$$\begin{cases} \cos \alpha = \dfrac{S_x}{|\vec{S}|} = \dfrac{4}{5}, \\ \cos \beta = \dfrac{S_y}{|\vec{S}|} = \dfrac{3}{5}. \end{cases}$$

$$\frac{\partial u}{\partial s} = u'_x \cdot \cos\alpha + u'_y \cos\beta \; = $$

$$\begin{cases} u'_x = 2x, \\ u'_y = y \end{cases} \qquad \begin{cases} u'_x \big|_{P_0(2,-1)} = 4, \\ u'_y \big|_{P_0(2,-1)} = -1 \end{cases}$$

$$= 4 \cdot \frac{4}{5} - 1 \cdot \frac{3}{5} = \frac{16-3}{5} = \frac{13}{5}$$ Ответ

10.33 $u = x_1^2 + x_2^2 - x_3^2 + x_4^2$

$P_0 = (1, 3, 2, -1)$

$\bar{a} = 2\bar{e}_1 + \vec{e}_2 - 2\bar{e}_4$ – направляющий вектор

$\frac{\partial u}{\partial \vec{a}}$ – ?

$\vec{a} = \{2, 1, 0, -2\}$

$|\bar{a}| = \sqrt{4+1+4} = 3$

$\cos\alpha = \frac{2}{3}$

$\cos\beta = \frac{1}{3}$

$\cos\gamma = 0$

$\cos\Theta = -\frac{2}{3}$

$u'_{x_1} = 2x_1$

$u'_{x_2} = 2x_2$

$u'_{x_3} = -2x_3$

$u'_{x_4} = 2x_4$

$u'_{x_1}\big|_{P_0} = 2$

$u'_{x_2}\big|_{P_0} = 6$

$u'_{x_3}\big|_{P_0} = -4$

$u'_{x_4}\big|_{P_0} = -2$

$$\frac{\partial u}{\partial \vec{a}} = u'_{x_1}\Big|_{P_0} \cdot \cos\alpha + u'_{x_2}\Big|_{P_0} \cdot \cos\beta + u'_{x_3}\Big|_{P_0} \cdot \cos\gamma +$$

$$+ u'_{x}\Big|_{P_0} \cdot \cos \theta$$

$$\frac{\partial u}{\partial \bar{a}} = 2 \cdot \frac{2}{3} + 6 \cdot \frac{1}{3} - 4 \cdot 0 - 2 \cdot \left(-\frac{2}{3}\right) =$$

$$= \frac{4}{3} + 2 + \frac{4}{3} = \frac{8}{3} + 2 = 2\frac{2}{3} + 2 = \frac{14}{3}$$

Ответ.

10.35 $u = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$, $P(a, b, c)$

$\frac{\partial u}{\partial \overline{OP}}$ - ?

$\overline{OP} = \{a, b, c\}$

$\parallel$

$\vec{S}$

$|\vec{S}| = \sqrt{a^2+b^2+c^2}$

$\cos\alpha = \frac{a}{\sqrt{a^2+b^2+c^2}}$

$\cos\beta = \frac{b}{\sqrt{a^2+b^2+c^2}}$

$\cos\gamma = \frac{c}{\sqrt{a^2+b^2+c^2}}$

$\frac{\partial u}{\partial x} = \frac{2x}{a^2}$

$\frac{\partial u}{\partial y} = \frac{2y}{b^2}$

$\frac{\partial u}{\partial z} = \frac{2z}{c^2}$

$\frac{\partial u}{\partial x}\Big|_{P_0} = \frac{2}{a}$

$\frac{\partial u}{\partial y}\Big|_{P_0} = \frac{2}{b}$

$\frac{\partial u}{\partial z}\Big|_{P_0} = \frac{2}{c}$

$$\frac{\partial u}{\partial s} = u'_x\big|_{P_0} \cos\alpha + u'_y\big|_{P_0} \cdot \cos\beta + u'_z\big|_{P_0} \cdot \cos\gamma =$$

$$= \frac{2}{\cancel{a}} \cdot \frac{\cancel{a}}{\sqrt{a^2+b^2+c^2}} + \frac{2}{\cancel{b}} \cdot \frac{\cancel{b}}{\sqrt{a^2+b^2+c^2}} +$$

$$+ \frac{2}{\cancel{c}} \cdot \frac{\cancel{c}}{\sqrt{a^2+b^2+c^2}} =$$

$$= \frac{6}{\sqrt{a^2+b^2+c^2}}$$ Ответ

10.37 $u = xyz$, $P_0(1, 2, 2)$

grad u – ?

$u'_x = yz$	$u'_x\big|_{P_0} = 4$
$u'_y = xz$	$u'_y\big|_{P_0} = 2$
$u'_z = xy$	$u'_z\big|_{P_0} = 2$

$$\text{grad } u = \nabla u = \{4, 2, 2\} = 4\cdot\vec{i} + 2\cdot\vec{j} + 2\vec{k}.$$

10.39 $u = 2x^2 - 4xy + y^2 - 2yz + 6z$

стац. точки – ?

Стац. точки – точки, в которых производная ф-ции в $\forall$ направлении равно 0.

$$\frac{\partial u}{\partial s} = u'_x \cos\alpha + u'_y \cos\beta + u'_z \cdot \cos\gamma = 0$$

при $\forall$ направлении $\Rightarrow$ $\left.\begin{matrix}\cos\alpha \\ \cos\beta \\ \cos\gamma\end{matrix}\right\} \forall \Rightarrow$

$$\Rightarrow \begin{cases} u'_x = 0, \\ u'_y = 0, \\ u'_z = 0 \end{cases} \Rightarrow \begin{cases} 4x - 4y = 0, \\ -4x + 2y - 2z = 0, \\ -2y + 6 = 0. \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x = y, \\ -2y = -6, \\ 2z = 2y = -4 \end{cases} \Rightarrow \begin{cases} y = 3, \\ x = 3, \\ z = y - 2x = 3 - 6 = -3 \end{cases} \Rightarrow$$

$\Rightarrow (3, 3, -3)$ – стац. точка.