С.з №4. Тема: Производная сложной ф-ции. Производная неявно заданной ф-ции. Производная по направлению. Градиент.

Производная сложной функции:

$z = f(x, y)$ – диф. ф-ция 2-х переменных x и y

$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}, \quad \Rightarrow \quad z = f(\varphi(t), \psi(t)) \text{ – сложная функция}$$

ф-ция одной переменной

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}} \quad (1)$$

$z = f(x, y)$ – диф. ф-ция 2-х переменных x и y

$$y = y(x) \quad \Rightarrow \quad z = f(x, y(x)) \text{ – сложная функция}$$

ф-ция одной переменной

$$\boxed{\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx}} \quad (2)$$

$z = f(u, v)$ – диф-ма ф-ция 2-х переменных u и v, $\begin{cases} u = \varphi(x, y) \\ v = \psi(x, y) \end{cases}, \quad \Rightarrow$

$$\Rightarrow \boxed{z = f(\varphi(x,y), \psi(x,y))}$$ – сложная ф-ция

ф-ция 2-х переменных x и y

$$\begin{cases} \dfrac{\partial z}{\partial x} = \dfrac{\partial f}{\partial u}\cdot\dfrac{\partial u}{\partial x} + \dfrac{\partial f}{\partial v}\cdot\dfrac{\partial v}{\partial x}, \\ \dfrac{\partial z}{\partial y} = \dfrac{\partial f}{\partial u}\cdot\dfrac{\partial u}{\partial y} + \dfrac{\partial f}{\partial v}\cdot\dfrac{\partial v}{\partial y} \end{cases} \quad (3)$$

Задачи: 7.114, 7.119, 7.122, 7.129, 7.135 [Е.Д]

Д/З: 7.116, 7.118, 7.123, 7.130, 7.136

7.114

$$z = e^{2x-3y}, \quad x = tg\,t, \quad y = t^2 - t$$

$$\left.\begin{matrix} z = f(x,y) \\ x = x(t) \\ y = y(t) \end{matrix}\right\} \Rightarrow f(x(t), y(t)) = z$$

сложная ф-ция одной переменной t

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}} \quad -(1)$$

$$\frac{dz}{dt} = e^{2x-3y} \cdot 2 \cdot \frac{1}{\cos^2 t} + e^{2x-3y} \cdot (-3) \cdot (2t-1) =$$

$$= e^{2x-3y} \left(\frac{2}{\cos^2 t} - 6t + 3 \right)$$

$$\frac{\partial z}{\partial x} = 2 \cdot e^{2x-3y} \qquad \frac{dx}{dt} = \frac{1}{\cos^2 t}$$

$$\frac{\partial z}{\partial y} = -3 \cdot e^{2x-3} \qquad \frac{dy}{dt} = 2t-1$$

7.119 $\quad z = \operatorname{arctg} \dfrac{x-1}{y}$

$$y = e^{(x+1)^2}$$

$$\left. \begin{array}{l} z = f(x,y) \\ y = y(x) \end{array} \right\} \Rightarrow z = f(x, y(x))$$ – сложная ф-ция одной переменной

$$\boxed{\frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}}$$

$$\frac{\partial f}{\partial x} = \frac{1}{1 + \left(\frac{x-1}{y} \right)^2} \cdot \frac{1}{y} = \frac{y^{\cancel{2}}}{y^2 + (x-1)^2} \cdot \frac{1}{\cancel{y}} =$$

$$= \frac{y}{y^2 + (x-1)^2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{x-1}{y}\right)^2} \cdot (x-1) \cdot \left(-\frac{1}{y^2}\right) =$$

$$= \frac{1-x}{y^2+(x-1)^2} \cdot \frac{y^2}{y^2} = \frac{1-x}{y^2+(x-1)^2}$$

$$\frac{dy}{dx} = 2(x+1)e^{(x+1)^2}$$

$$\frac{dz}{dx} = \frac{y}{y^2+(x-1)^2} + \frac{1-x}{y^2+(x-1)^2} \cdot 2(x+1)e^{(x+1)^2} =$$

$$= \frac{1}{y^2+(x-1)^2}\left(y + 2(1-x)(x+1)e^{(x+1)^2}\right) =$$

$$= \frac{1}{y^2+(x-1)^2}\left(y - 2(x^2-1)e^{(x+1)^2}\right)$$

7.122

$$z = f(u, v)$$

$$u = \frac{2y}{x+y}$$

$$v = x^2 - 3y$$

Сложная ф-ция: $z = f(u(x,y), v(x,y))$

ф-ция 2-х переменных

x и y.

$$\begin{cases} \dfrac{\partial z}{\partial x} = \dfrac{\partial f}{\partial u}\cdot\dfrac{\partial u}{\partial x} + \dfrac{\partial f}{\partial v}\cdot\dfrac{\partial v}{\partial x}, \\ \dfrac{\partial z}{\partial y} = \dfrac{\partial f}{\partial u}\cdot\dfrac{\partial u}{\partial y} + \dfrac{\partial f}{\partial v}\cdot\dfrac{\partial v}{\partial y} \end{cases}$$

$$u'_x = -\frac{2y}{(x+y)^2}, \quad v'_x = 2x$$

$$u'_y = 2\,\frac{x+y-2y}{(x+y)^2} = 2\cdot\frac{x-y}{(x+y)^2}, \quad v'_y = -3$$

$$\begin{cases} \dfrac{\partial z}{\partial x} = f'_u\cdot\left(-\dfrac{2y}{(x+y)^2}\right) + f'_v\cdot 2x \\ \dfrac{\partial z}{\partial y} = f'_u\cdot 2\cdot\dfrac{x-y}{(x+y)^2} + f'_v\cdot(-3) \end{cases}$$

$$\begin{cases} \dfrac{\partial z}{\partial x} = -\dfrac{2y}{(x+y)^2}f'_u + 2x f'_v \\ \dfrac{\partial z}{\partial y} = 2\,\dfrac{x-y}{(x+y)^2}f'_u - 3f'_v \end{cases}$$

7.129 $\quad z = x \cdot f\left(\underbrace{\frac{y}{x}}_{=t}\right) - x^2 - y^2$

Док-ть равенство: $x z'_x + y z'_y = z - x^2 - y^2$

$$z'_x = f\left(\frac{y}{x}\right) + x \cdot f'\left(\frac{y}{x}\right) \cdot \left(\frac{y}{x}\right)'_x - 2x =$$

$$= f + x f' \cdot y \cdot \left(-\frac{1}{x^2}\right) - 2x =$$

$$= f - \frac{y}{x} f' - 2x$$

$$z'_y = x \cdot f'\left(\frac{y}{x}\right) \cdot \left(\frac{y}{x}\right)'_y - 2y =$$

$$= x \cdot \frac{1}{x} \cdot f' - 2y = f' - 2y$$

$$xf \cancel{- x \cdot \frac{y}{x} f'} - 2x \cdot x \cancel{+ y f'} - 2y \cdot y =$$

$$= z - x^2 - y^2$$

$$xf - 2x^2 - 2y^2 = z - x^2 - y^2 =$$

$$= xf - x^2 - y^2 - x^2 - y^2 =$$

$$= xf - 2x^2 - 2y^2$$

ч.т.д.

7.135 $u = x\varphi(\overbrace{x+y}^{=t}) + y\psi(\overbrace{x+y}^{=t})$

Док-ть рав-во: $u''_{xx} - 2u''_{xy} + u''_{yy} = 0$

$u'_x = \varphi(x+y) + x\cdot\varphi'(x+y) + y\psi'(x+y)$

$u''_{xx} = \varphi'(x+y) + \varphi'(x+y) + x\cdot\varphi''(x+y) + y\psi''(x+y) =$

$= 2\varphi'(x+y) + x\varphi''(x+y) + y\psi''(x+y)$

$u'_y = x\varphi'(x+y) + \psi(x+y) + y\psi'(x+y)$

$u''_{yy} = x\varphi''(x+y) + \psi'(x+y) + \psi'(x+y) + y\psi''(x+y) =$

$= x\varphi''(x+y) + 2\psi'(x+y) + y\psi''(x+y)$

$u''_{xy} = \left(\varphi(x+y) + x\varphi'(x+y) + y\psi'(x+y)\right)'_y =$

$= \varphi' + x\varphi'' + \psi' + y\psi''$

$\underbrace{\cancel{2\varphi'} + \cancel{x\varphi''} + \cancel{y\psi''}}_{u''_{xx}} \underbrace{- \cancel{2x\varphi''} - \cancel{2\varphi'} - \cancel{2\psi'} - \cancel{2y\psi''}}_{-2u''_{xy}} +$

$+ \underbrace{\cancel{x\varphi''} + \cancel{2\psi'} + \cancel{y\psi''}}_{u''_{yy}} = 0$

$0 = 0$

Производная неявно заданной функции

$f(x,y)=0$

$y=\varphi(x)$

$f(x,\varphi(x))=0$

$$\boxed{y'_x=-\frac{f'_x}{f'_y}}$$

$f'_y\neq 0.$

$F(x,y,z)=0$

$z=f(x,y)$

$$\boxed{z'_x=-\frac{F'_x(x,y,z)}{F'_z(x,y,z)}}$$

$$\boxed{z'_y=-\frac{F'_y(x,y,z)}{F'_z(x,y,z)}}$$

$F'_z(x,y,z)\neq 0$

Задачи: 7.141, 7.145, 7.149, 7.152

Д/З 7.140, 7.146, 7.150, 7.151.

7.141 $y\sin x-\cos(x-y)=0$

$y'_x - ?$

$f(x,y)=y\sin x-\cos(x-y)=0$

$$y'_x=-\frac{f'_x}{f'_y}=-\frac{y\cos x+\sin(x-y)}{\sin x+\cos(x-y)}$$

7.145 $z^3 - 4xz + y^2 - 4 = 0.$

$M(1, -2, 2)$

$F(x, y, z) = 0$

$f(x, y) = z$

$$\begin{cases} z'_x = -\dfrac{F'_x}{F'_z} = -\dfrac{-4z}{3z^2 - 4x}, \\ z'_y = -\dfrac{F'_y}{F'_z} = -\dfrac{2y}{3z^2 - 4x} \end{cases}$$

$$\begin{cases} F'_x = -4z, \\ F'_y = 2y, \\ F'_z = 3z^2 - 4x. \end{cases}$$

$$z'_x = \frac{4z}{3z^2 - 4x}\Bigg|_{M(1,-2,2)} = \frac{4 \cdot 2}{3 \cdot 4 - 4 \cdot 1} = \frac{8}{12 - 4} = \frac{8}{8} = 1$$

$$z'_y = -\frac{2y}{3z^2 - 4x}\Bigg|_{M(1,-2,2)} = \frac{-2 \cdot (-2)}{3 \cdot 4 - 4 \cdot 1} = \frac{4}{12 - 4} = \frac{4}{8} = \frac{1}{8}$$

7.149

$yz = \operatorname{arctg}(xz)$ $\qquad dz - ?$

$yz - \operatorname{arctg}(xz) = 0$

$F(x,y,z) = 0$

$\boxed{F(x,y,z) = yz - \operatorname{arctg}(xz) = 0}$

$z = f(x,y)$

$\boxed{dz = z'_x\,dx + z'_y\,dy}$

$$\begin{cases} z'_x = -\dfrac{F'_x}{F'_z}, \\ z'_y = -\dfrac{F'_y}{F'_z} \end{cases}$$

$$\begin{cases} F'_x = -\dfrac{1}{1+x^2z^2}\cdot z \\ F'_y = z \\ F'_z = y - \dfrac{1}{1+x^2z^2}\cdot x \end{cases}$$

$$dz = \left(\frac{x}{1+x^2z^2} - y\right)\left(-\frac{z}{1+x^2z^2}dx + z\,dy\right)$$

7.152

$$x+y+z=e^z \qquad z''_{xx}-?$$

$$x+y+z-e^z=0 \qquad z''_{xy}-?$$

$$\boxed{F(x,y,z)=x+y+z-e^z} \qquad z''_{yy}-?$$

$$z=f(x,y)$$

$$\begin{cases} F'_x=1, \\ F'_y=1, \\ F'_z=1-e^z \end{cases} \qquad z'_x=-\frac{F'_x}{F'_z}=-\frac{1}{1-e^z}$$

$$z'_y=-\frac{F'_y}{F'_z}=-\frac{1}{1-e^z}$$

$$z''_{xx}=\left(\frac{1}{e^z-1}\right)'_x=-\frac{1\cdot e^z}{(e^z-1)^2}\cdot z'_x=-\frac{1\cdot e^z}{(e^z-1)^2}\cdot -\frac{1}{1-e^z}=$$

$$=-\frac{1\cdot e^z}{(e^z-1)^2}\cdot\frac{1}{(e^z-1)}=-\frac{1\cdot e^z}{(e^z-1)^3}$$

$$z''_{yy}=-\frac{e^z}{(e^z-1)^3}$$

$$z''_{xy}=-\frac{e^z}{(e^z-1)^3}$$