Приложения квадратичных форм

Вариант 9

а)
$$12xy - 9x^2 - 4y^2 + 14\sqrt{13}\,x - 18\sqrt{13}\,y - 65 = 0 \quad (1)$$

Матрица КФ: $A = \begin{pmatrix} -9 & 6 \\ 6 & -4 \end{pmatrix}$

Найдём собств. числа: $(9+\lambda)(4+\lambda) - 36 = 0$

$$\lambda^2 + 13\lambda = 0 \Rightarrow \left[\begin{array}{l} \lambda_1 = -13 \\ \lambda_2 = 0 \end{array}\right.$$

Найдём собств. вектора:

$$\begin{pmatrix} -9-\lambda & 6 \\ 6 & -4-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -(9+\lambda)x_1 + 6y_1 = 0 \\ 6x_1 - (4\lambda)y_1 = 0 \end{cases}$$

$\lambda_2 = -13$: $\begin{cases} 4x_1 + 6y_1 = 0 \\ 6x_1 + 9y_1 = 0 \end{cases} \rightarrow 2x_1 + 3y_1 = 0$

$$\begin{cases} x_1 = -3c \\ y_1 = 2c \end{cases}, \; c \in R \setminus \{0\}$$

$$\bar{e}_1 = \left(\frac{-3}{\sqrt{13}}; \frac{2}{\sqrt{13}}\right)^T$$

$\lambda_2 = 0$: $\begin{cases} -9x_2 + 6y_2 = 0 \\ 6x_2 - 4y_2 = 0 \end{cases} \rightarrow -3x_2 + 2y_2 = 0$

$$\begin{cases} x_2 = 2c \\ y_2 = 3c \end{cases}$$

$$\bar{e}_2 = \left(\frac{2}{\sqrt{13}}; \frac{3}{\sqrt{13}}\right)^T$$

$$U = \frac{1}{\sqrt{13}}\begin{pmatrix} -3 & 2 \\ 2 & 3 \end{pmatrix}; \quad \begin{cases} x^* = \frac{-3}{\sqrt{13}}x' + \frac{2}{\sqrt{13}}y' \\ y^* = \frac{2}{\sqrt{13}}x' + \frac{3}{\sqrt{13}}y' \end{cases}$$

Подставив в (1) получим: $-13x'^2 - 78x' - 26y' - 65 = 0$

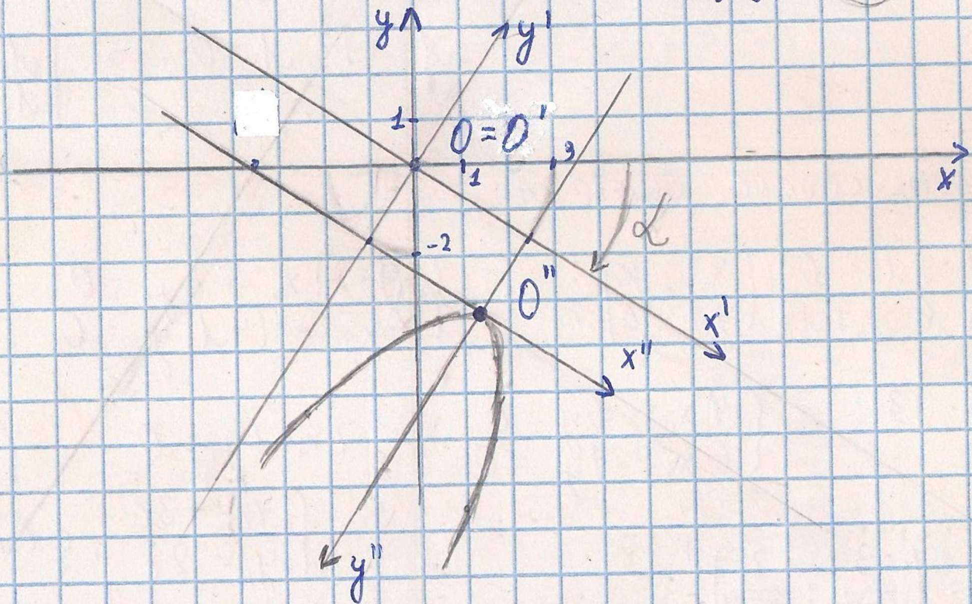
$$(x' - 3)^2 = 2(-y - 2)$$

$(x'-3)^2 = 2(-y-2)$ - ПАРАБОЛА СО СМЕЩЕНИЕМ

$\begin{cases} x'' = x' - 3 \\ y'' = -y' - 2 \end{cases} \to (x'')^2 = 2(y'')$ - КАНОНИЧ. УР-Е ПАРАБОЛЫ

$\alpha = \operatorname{arctg}\left(\frac{-2\sqrt{13}}{3\sqrt{13}}\right) = -\operatorname{arctg}\frac{2}{3} \approx 63^0$

$\begin{cases} x = \frac{-3}{\sqrt{13}}x' + \frac{2}{\sqrt{13}}y' = \frac{-3}{\sqrt{13}}x'' - \frac{2}{\sqrt{13}}y'' + \frac{5}{\sqrt{13}} \approx 1.3 \\ y = \frac{2}{\sqrt{13}}x' + \frac{3}{\sqrt{13}}y' = \frac{2}{\sqrt{13}}x'' - \frac{3}{\sqrt{13}}y'' - \frac{12}{\sqrt{13}} \approx -3.3 \end{cases}$ O''



б)

$3x^2 + 6y^2 - 4xy + 8\sqrt{5}x + 4\sqrt{5}y + 20 = 0$ (2)

$A = \begin{pmatrix} 3 & -2 \\ -2 & 6 \end{pmatrix}$; $(\lambda-3)(\lambda-6) - 4 = 0$
$\lambda^2 - 9\lambda + 14 = 0 \Rightarrow \left[\begin{array}{l} \lambda_1 = 2 \\ \lambda_2 = 7 \end{array}\right.$ - СОБСТВ. ЧИСЛА

Собств. вектора:

$\begin{cases} (3-\lambda)x_1 - 2y_1 = 0 \\ -2x_1 + (6-\lambda)y_1 = 0 \end{cases}$

$\lambda_1 = 2 : \begin{cases} x_1 - 2y_1 = 0 \\ -2x_1 - 4y_1 = 0 \end{cases} \to -x_1 + 2y_1 = 0 \to \begin{cases} x_1 = 2c \\ y_1 = c \end{cases}$

$\bar{e}_1 = \left(\frac{2}{\sqrt{5}}; \frac{1}{\sqrt{5}}\right)^T$

$\lambda_2 = 7 : \begin{cases} -4x_1 - 2y_1 = 0 \\ -2x_1 - y_1 = 0 \end{cases} \to 2x_1 + y_1 = 0 \to \begin{cases} x = -c \\ y = 2c \end{cases}$

$\bar{e}_2 = \left(\frac{-1}{\sqrt{5}}; \frac{2}{\sqrt{5}}\right)^T$

$$U = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}; \quad \begin{cases} x = \frac{2}{\sqrt{5}}x' - \frac{1}{\sqrt{5}}y' \\ y = \frac{1}{\sqrt{5}}x' - \frac{2}{\sqrt{5}}y' \end{cases}$$

Подставим в (2):

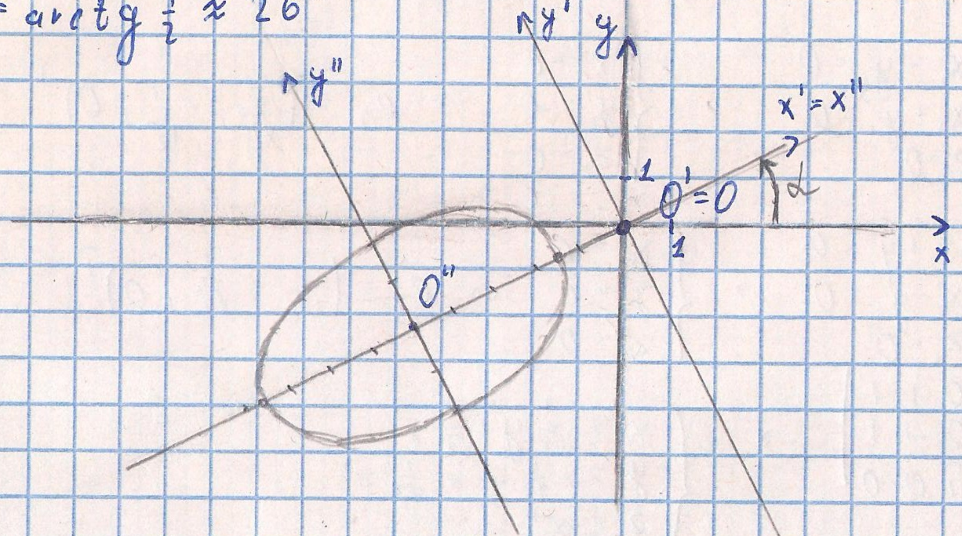
$$2(x')^2 + 7(y')^2 + 20x' + \cancel{20}\,22 = 0$$

$$\frac{(x'+5)^2}{14} + \frac{y'}{4} = 1$$ – ур-е эллипса со смещением

$$\begin{cases} x'' = x'+5 \\ y'' = y' \end{cases} \rightarrow \frac{(x'')^2}{14} + \frac{(y'')^2}{4} = 1$$ – каноннич. ур-е эллипса

$$\begin{cases} x = \frac{2}{\sqrt{5}}x' - \frac{1}{\sqrt{5}}y' = \frac{2}{\sqrt{5}}x'' - \frac{1}{\sqrt{5}} - \frac{10}{\sqrt{5}} \approx -4,4 \\ y = \frac{1}{\sqrt{5}}x' + \frac{2}{\sqrt{5}}y' = \frac{1}{\sqrt{5}}x'' + \frac{2}{\sqrt{5}}y'' - \frac{5}{\sqrt{5}} \approx -2,2 \end{cases} \quad O''$$

$\alpha = \operatorname{arctg}\frac{1}{2} \approx 26^0$



c) $x^2+y^2-2z^2+2xy+4\sqrt{2}x-4\sqrt{2}y+48-2=0$

$$A=\begin{pmatrix}1&1&0\\1&1&0\\0&0&-2\end{pmatrix};\quad |A-\lambda E|=0 \Rightarrow -(2+\lambda)\begin{vmatrix}1-\lambda&1\\1&1-\lambda\end{vmatrix}=0$$

$$(2+\lambda)((\lambda-1)^2-1)=0$$

$$\lambda(\lambda+2)(\lambda-2)=0 \rightarrow \left[\begin{array}{l}\lambda_1=-2\\\lambda_2=0\\\lambda_3=2\end{array}\right.$$

собств. числа

Найдём собств. вектора:

$$\begin{cases}(1-\lambda)x_1+y_1=0\\x_1+(1-\lambda)y_1=0\\-(2-\lambda)z_1=0\end{cases}$$

$$\lambda_1=-2:\ \left.\begin{cases}x_1+y_1=0\\x_1+\ \ y_1=0\\z_1=c\end{cases}\right]\rightarrow 4x_1+4y_1=0 \Rightarrow \begin{cases}x_1=0\\y_1=0\\z=c\end{cases}\rightarrow \bar{e}_1=(0;0;1)^T$$

$$\lambda_2=0:\ \begin{cases}x_1+y_1=0\\x_1+y_1=0\\z=0\end{cases}\rightarrow\begin{cases}x_1=c\\y_1=-c\\z_1=0\end{cases}\rightarrow \bar{e}_2=\left(\frac{1}{\sqrt{2}};-\frac{1}{\sqrt{2}};0\right)^T$$

$$\lambda_3=2:\ \begin{cases}-x_1+y_1=0\\x_1-y_1=0\\z_1=0\end{cases}\rightarrow\begin{cases}x=c\\y=c\\z=0\end{cases}\rightarrow \bar{e}_3=\left(\frac{1}{\sqrt{2}};\frac{1}{\sqrt{2}};0\right)^T$$

$$U=\frac{1}{\sqrt{2}}\begin{pmatrix}0&1&1\\0&-1&1\\\sqrt{2}&0&0\end{pmatrix}\rightarrow\begin{cases}x=\frac{1}{\sqrt{2}}y'+\frac{1}{\sqrt{2}}z'\\y=-\frac{1}{\sqrt{2}}y'+\frac{1}{\sqrt{2}}z'\\z=x'\end{cases}\quad 2b^TUX'=4x'+8y'$$

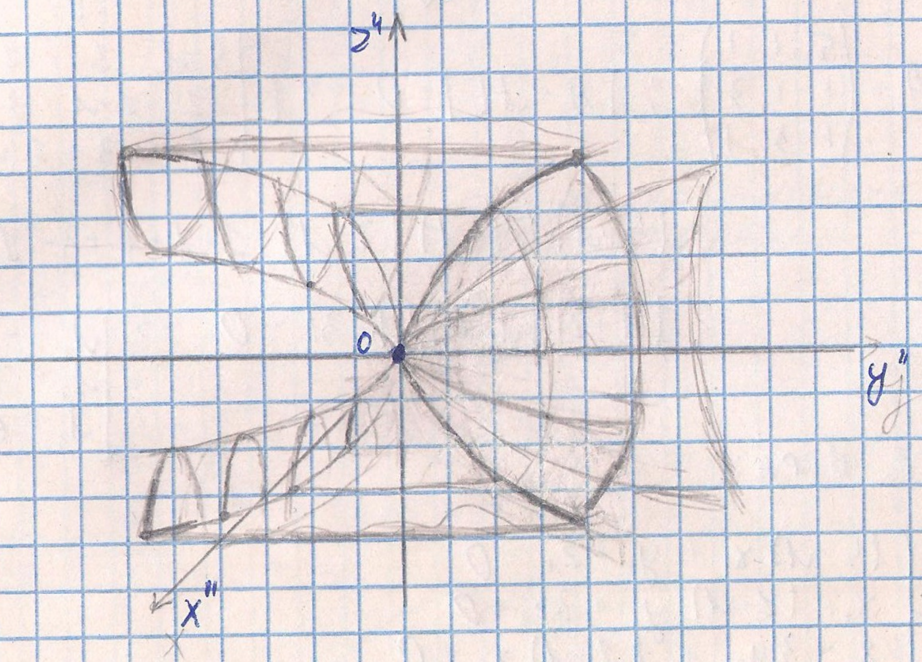
В итоге получим:

$$-2(x')^2+4x'+8y'+28z^2-2=0$$

$$\frac{(x'-1)^2}{2}-\frac{(z')^2}{2}=2y'$$

$$\begin{cases}x''=x'-1\\y''=y'\\z''=z'\end{cases}\rightarrow\frac{(x'')^2}{2}-\frac{(z'')^2}{2}=2y''$$ — гип. параболоид.

График в координатах (x'', y'', z''):



$x'' = 0 \to -\frac{(z'')^2}{2} = 2y''$ - ПАРАБОЛА

$z'' = 0 \to \frac{(x'')^2}{2} = 2y''$ - ПАРАБОЛА

$y'' = 0 \to \frac{(x'')^2}{2} - \frac{(z'')^2}{2} = 0$ - ГИПЕРБОЛА

d) $5x^2 + y^2 + z^2 + 2xy + 2xz + 6yz + 12\sqrt{6}x + 6\sqrt{6}y + 6\sqrt{6}z = 0$

$$A = \begin{pmatrix} 5 & 1 & 1 \\ 1 & 1 & 3 \\ 1 & 3 & 1 \end{pmatrix}; \quad |A - \lambda E| = 0 \Rightarrow \begin{vmatrix} 5-\lambda & 1 & 1 \\ 1 & 1-\lambda & 3 \\ 1 & 3 & 1-\lambda \end{vmatrix} = 0$$

$$-(5-\lambda)(4-\lambda)(2+\lambda) - 1 + \lambda + 3 + 3 - 1 + 1 = 0$$

$$(\lambda+2)(\lambda-6)(\lambda-3) = 0 \Rightarrow \left[\begin{array}{l} \lambda_1 = -2 \\ \lambda_2 = 3 \\ \lambda_3 = 6 \end{array}\right.$$

Собств. в-ра:

$$\begin{cases} (5-\lambda)x_1 + y_1 + z_1 = 0 \\ x_1 + (1-\lambda)y_1 + 3z_1 = 0 \\ x_1 + 3y_1 + (1-\lambda)z_1 = 0 \end{cases}$$

$\lambda_1 = -2:$ $\begin{cases} 7x_1 + y_1 + z_1 = 0 \\ x_1 + 3y_1 + 3z_1 = 0 \\ x_1 + 3y_1 + 3z_1 = 0 \end{cases} \rightarrow \begin{cases} x_1 = 0 \\ z_1 = -y_1 \end{cases} \Rightarrow \bar{e}_1\left(0; -\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}\right)$

$\lambda_2 = 3:$ $\begin{cases} 2x_1 + y_1 + z_1 = 0 \\ x_1 - 2y_1 + 3z_1 = 0 \\ x_1 - 2y_1 + 3z_1 = 0 \end{cases} \rightarrow \begin{cases} y_1 - z_1 = 0 \\ x_1 = -y_1 \end{cases} \Rightarrow \bar{e}_2\left(\frac{-1}{\sqrt{3}}; \frac{1}{\sqrt{3}}; \frac{1}{\sqrt{3}}\right)$

$\lambda_3 = 6:$ $\begin{cases} -x_1 + y_1 + z_1 = 0 \\ x_1 - 5y_1 + 3z_1 = 0 \\ x_1 + 3y_1 - 5z_1 = 0 \end{cases} \rightarrow \begin{cases} -x - 2y_1 = 0 \\ y_1 - 2z_1 = 0 \end{cases} \Rightarrow \bar{e}_3\left(\frac{2}{\sqrt{6}}; \frac{1}{\sqrt{6}}; \frac{1}{\sqrt{6}}\right)$

$$U = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & -\sqrt{2} & 2 \\ -\sqrt{3} & \sqrt{2} & 1 \\ \sqrt{3} & \sqrt{2} & 1 \end{pmatrix}; \quad \begin{cases} x = -\frac{1}{\sqrt{3}}y_1 + \frac{2}{\sqrt{6}}z_1 \\ y = -\frac{1}{\sqrt{2}}x_1 + \frac{1}{\sqrt{3}}y_1 + \frac{1}{\sqrt{6}}z_1 \\ z = \frac{1}{\sqrt{2}}x_1 + \frac{1}{\sqrt{3}}y_1 + \frac{1}{\sqrt{6}}z_1 \end{cases}$$

$2b^T U X_1 = 36z$

В итоге получим:

$$-2(x')^2 + 3(y')^2 + 6(z')^2 + 36z = 0$$

$$-2(x')^2 + 3(y')^2 + 6(z'+3)^2 = 54 \quad | : 54$$

$$\begin{cases} x' = x'' \\ y' = y'' \\ z' + 3 = z'' \end{cases} \to -\frac{(x'')^2}{27} + \frac{(y'')^2}{18} + \frac{(z'')^2}{9} = 1$$

УР-Е ОДНОПОЛОСТНОГО ГИПЕРБОЛОИДА

$x'' = 0 \to \frac{y''}{18} + \frac{z''}{9} = 1$ – ЭЛЛИПС

$y'' = 0 \to \frac{z''}{9} - \frac{x''}{21} = 1$ – ГИПЕРБОЛА

$z'' = 0 \to \frac{y''}{18} - \frac{x''}{27} = 1$ – ГИПЕРБОЛА