Примерный вариант билета РК №2

по ЛА и ФНП

Часть А

4) Составить ур-ние касательной плоскости и нормали к поверхности $z = x - \sqrt{x^2+y^2}$ в точке $(4, 3, -1)$.

Решение: $z = x - \sqrt{x^2+y^2}$

$$\begin{cases} z'_x = 1 - \dfrac{1 \cdot \cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}} = 1 - \dfrac{x}{\sqrt{x^2+y^2}} \\ z'_y = 1 - \dfrac{1 \cdot \cancel{2}y}{\cancel{2} \cdot \sqrt{x^2+y^2}} = 1 - \dfrac{y}{\sqrt{x^2+y^2}} \end{cases}$$

$$z'_x \Big|_{M(4,3,-1)} = 1 - \frac{4}{5} = \frac{1}{5}$$

$$z'_y \Big|_{M(4,3,-1)} = -\frac{3}{5}$$

$$z + 1 = \frac{1}{5}(x-4) - \frac{3}{5}(y-3)$$

$$z + 1 = \frac{x}{5} - \frac{4}{5} - \frac{3}{5}y + \frac{9}{5}$$

$$z - \frac{x}{5} + \frac{3}{5}y - \cancel{1} + \cancel{1} = 0$$

$\boxed{5z - x + 3y = 0}$ уравнение касательная плоскость

$\boxed{\dfrac{5(x-4)}{1} = \dfrac{5(y-3)}{-3} = \dfrac{z+1}{1}}$ уравнение нормали

5 Исследовать на экстремум ф-цию
$z = e^{2x} + e^{2y} - x - y$

Решение: $z = e^{2x} + e^{2y} - x - y$

$$\begin{cases} z'_x = 2e^{2x} - 1 = 0, \\ z'_y = 2e^{2y} - 1 = 0 \end{cases} \Rightarrow \begin{cases} 2e^{2x} = 1, \\ 2e^{2y} = 1 \end{cases} \Rightarrow$$

$$\begin{cases} 2e^{2x} = 1, \\ 2e^{2y} = 1 \end{cases} \Rightarrow \begin{cases} e^{2x} = \frac{1}{2}, \\ e^{2y} = \frac{1}{2} \end{cases} \Rightarrow \begin{cases} 2x = \ln\frac{1}{2}, \\ 2y = \ln\frac{1}{2} \end{cases} \Rightarrow$$

$$\begin{cases} x = \frac{1}{2}\ln\frac{1}{2}, \\ y = \frac{1}{2}\ln\frac{1}{2} \end{cases} \Rightarrow P\left(\frac{1}{2}\ln\frac{1}{2}; \frac{1}{2}\ln\frac{1}{2}\right)$$

$$\begin{cases} z''_{xx} = A = 4e^{2x} \\ z''_{xy} = B = 0 \\ z''_{yy} = C = 4e^{2y} \end{cases} \Rightarrow \quad A|_P = 4e^{\cancel{2}\cdot\frac{1}{\cancel{2}}\ln\frac{1}{2}} = 4\cdot\frac{1}{2} = 2 \quad C|_P = 4\cdot e^{\cancel{2}\cdot\frac{1}{\cancel{2}}\ln\frac{1}{2}} = 4\cdot\frac{1}{2} = 2$$

$D = AC - B^2 = 4 - 0 = 4 > 0 \Rightarrow \exists$ экстремум

$A > 0 \Rightarrow$ минимум

$P\left(\frac{1}{2}\ln\frac{1}{2}; \frac{1}{2}\ln\frac{1}{2}\right)$ – точка минимума

$$z|_P = e^{\cancel{2}\cdot\frac{1}{\cancel{2}}\ln\frac{1}{2}} + e^{\cancel{2}\cdot\frac{1}{\cancel{2}}\ln\frac{1}{2}} - \frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}\ln\frac{1}{2} =$$

$$= 1 - \ln\frac{1}{2} = 1 - \ln 1 + \ln 2 = 1 + \ln 2$$

Ответ: $z_{min} = 1 + \ln 2$

⑥ Исследовать на экстремум ф-цию $z = e^{-2xy}$ при условии $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Решение:

$$F(x, y, \lambda) = e^{-2xy} + \lambda\left(\frac{x^2}{9} + \frac{y^2}{4} - 1\right)$$

$$\begin{cases} F'_x = -2y e^{-2xy} + \frac{2}{9} x \lambda = 0, \\ F'_y = -2x e^{-2xy} + \frac{2}{4} y \lambda = 0, \\ F'_\lambda = \frac{x^2}{9} + \frac{y^2}{4} - 1 = 0. \end{cases}$$

$F''_{xx} = \frac{2}{9}\lambda + 4y^2 e^{-2xy}$

$F''_{xy} = -2e^{-2xy} + 4xy e^{-2xy}$

$F''_{yy} = 4x^2 e^{-2xy} + \frac{2}{4}\lambda$

$$\begin{cases} y e^{-2xy} - \frac{1}{9} x\lambda = 0, & \Big| \; x \\ x e^{-2xy} - y\lambda = 0, & \Big| \; y \quad \text{"}-\text{"} \Rightarrow \\ \frac{x^2}{9} + \frac{y^2}{4} - 1 = 0. \end{cases}$$

$\frac{1}{9} x\lambda = y e^{-2xy}$

$\lambda = 9 \frac{y}{x} e^{-2xy}$

$-\frac{1}{9}x^2\lambda + y^2\lambda = 0$

$\lambda\left(y^2 - \frac{1}{9}x^2\right) = 0$

$\lambda = 0 \qquad y^2 - \frac{1}{9}x^2 = 0 \Rightarrow \left(y - \frac{1}{3}x\right)\left(y + \frac{1}{3}x\right) = 0$

$y = \frac{1}{3}x$

$y = -\frac{1}{3}x$

$$\begin{vmatrix} 0 & \frac{2}{9}x & \frac{1}{2}y \\ \frac{2}{9}x & F''_{xx} & F''_{xy} \\ \frac{1}{2}y & F''_{xy} & F''_{yy} \end{vmatrix} =$$

$= \frac{2}{9}xy F''_{xy} - \frac{1}{4}y^2 F''_{xx} - \frac{4}{81}x^2 F''_{yy}$

$\frac{1}{9}x^2 + \frac{1}{4}\cdot\frac{1}{9}x^2 - 1 = 0$

$$\frac{5}{9}x^2 = 1$$

$$x^2 = \frac{9}{5}$$

$$x = \pm \frac{3\sqrt{5}}{5}$$

$y = \frac{1}{3}x$: $x = \frac{3\sqrt{5}}{5}$; $x = -\frac{3\sqrt{5}}{5}$, $\lambda = 3e^{-\frac{6}{5}}$

$y = \frac{\sqrt{5}}{5}$; $y = -\frac{\sqrt{5}}{5}$

$y = -\frac{1}{3}x$: $x = \frac{3\sqrt{5}}{5}$; $x = -\frac{3\sqrt{5}}{5}$, $\lambda = -3e^{\frac{6}{5}}$

$y = -\frac{\sqrt{5}}{5}$; $y = \frac{\sqrt{5}}{5}$

Стац. точки:

$$\left.\begin{matrix} P_1\left(\frac{3\sqrt{5}}{5}; \frac{\sqrt{5}}{5}\right) \\ P_2\left(-\frac{3\sqrt{5}}{5}; -\frac{\sqrt{5}}{5}\right)\end{matrix}\right\} y = \frac{1}{3}x$$

$$\left.\begin{matrix} P_3\left(\frac{3\sqrt{5}}{5}; -\frac{\sqrt{5}}{5}\right) \\ P_4\left(-\frac{3\sqrt{5}}{5}; \frac{\sqrt{5}}{5}\right)\end{matrix}\right\} y = -\frac{1}{3}x$$

$$F''_{xx} = -2y^2e^{-2xy} - \frac{1}{9}\lambda$$

$$F''_{xy} = e^{-2xy} - 2xye^{-2xy}$$

$$F''_{yy} = -2x^2e^{-2xy} - \lambda$$

$\underline{P_1\left(\frac{3\sqrt{5}}{5};\frac{\sqrt{5}}{5}\right)}$; $\underline{P_2\left(-\frac{3\sqrt{5}}{5};-\frac{\sqrt{5}}{5}\right)}$ условный max

$\underline{\lambda = 3e^{-\frac{6}{5}}}$; $\underline{\lambda = 3e^{-\frac{6}{5}}}$

$$F''_{xx} = -2\cdot\frac{1}{5}e^{-2\cdot\frac{3\sqrt{5}}{5}\cdot\frac{\sqrt{5}}{5}} - \frac{1}{9}\cdot 3e^{-\frac{6}{5}} = -\frac{2}{5}e^{-\frac{6}{5}} - \frac{1}{3}e^{-\frac{6}{5}} =$$

$$= \frac{-6-5}{15}e^{-\frac{6}{5}} = -\frac{11}{15}e^{-\frac{6}{5}}$$

$$F''_{xy} = e^{-\frac{6}{5}} - \frac{6}{5}e^{-\frac{6}{5}} = -\frac{1}{5}e^{-\frac{6}{5}}$$

$$F''_{yy} = -2\cdot\frac{9}{5}e^{-\frac{6}{5}} - 3e^{-\frac{6}{5}} = -\frac{18}{5}e^{-\frac{6}{5}} - 3e^{-\frac{6}{5}} =$$

$$= \frac{-18-15}{5}e^{-\frac{6}{5}} = -\frac{33}{5}e^{-\frac{6}{5}}$$

Уравнение связи: $\frac{x^2}{9} + \frac{y^2}{4} = 1$

$$\frac{2}{9}x\,dx + \frac{2}{4}y\,dy$$

$$\frac{1}{2}y\,dy = -\frac{2}{9}x\,dx$$

$$\boxed{dy = -\frac{4}{9}\cdot\frac{x}{y}dx}$$

$$dy = -\frac{4}{\cancel{9}\,3}\cdot\frac{\cancel{3}\cancel{\sqrt{5}}\cdot\cancel{5}}{\cancel{5}\cdot\cancel{\sqrt{5}}}dx = \boxed{-\frac{4}{3}dx = dy}$$

$$\boxed{d^2F = F''_{xx}dx^2 + 2F''_{xy}dx\,dy + F''_{yy}dy^2}$$

$$d^2F = -\frac{11}{15}e^{-\frac{6}{5}}dx^2 + 2\cdot\left(-\frac{1}{5}\right)e^{-\frac{6}{5}}dx\left(-\frac{4}{3}\right)dx -$$

$$-\frac{33}{5}e^{-\frac{6}{5}}\left(-\frac{4}{3}\right)^2dx^2 = \left(-\frac{11}{15} + \frac{8}{15} - \frac{16\cdot 33}{9\cdot 5}\right)dx^2 =$$

$$= \left(\frac{3}{15} - \frac{528}{45}\right)dx^2 = \frac{27-528}{45}dx^2 = \frac{-501}{45}dx^2 < 0$$

$P_3\left(\frac{3\sqrt{5}}{5}; -\frac{\sqrt{5}}{5}\right)$; $P_4\left(-\frac{3\sqrt{5}}{5}; \frac{\sqrt{5}}{5}\right)$ условный минимум

$\lambda = -3e^{\frac{6}{5}}$ $\qquad$ $\lambda = -3e^{\frac{6}{5}}$

$$F''_{xx} = -2\cdot\frac{1}{5}e^{\frac{6}{5}} + \frac{1}{9}\cdot 3e^{\frac{6}{5}} = 3e^{\frac{6}{5}} - \frac{2}{5}e^{\frac{6}{5}} = \frac{13}{5}e^{\frac{6}{5}}$$

$$F''_{xy} = e^{\frac{6}{5}} + \frac{6}{5}e^{\frac{6}{5}} = \frac{11}{5}e^{\frac{6}{5}}$$

$$F''_{yy} = -2\cdot\frac{9}{5}e^{\frac{6}{5}} + 3e^{\frac{6}{5}} = -\frac{18}{5}e^{\frac{6}{5}} + 3e^{\frac{6}{5}} =$$

$$= \frac{-18+15}{5}e^{\frac{6}{5}} = -\frac{3}{5}e^{\frac{6}{5}}$$

Уравнение связи: $dy = -\frac{4}{9}\cdot\frac{x}{y}dx$

$$dy = -\frac{4}{9}\cdot\frac{3\sqrt{5}\cdot 5}{5\cdot(-\sqrt{5})}dx =$$

$$= \frac{4}{3}dx = dy$$

$$d^2F = \frac{13}{5}e^{\frac{6}{5}}dx^2 + 2\cdot\frac{11}{5}e^{\frac{6}{5}}dx\cdot\frac{4}{3}dx - \frac{3}{5}e^{\frac{6}{5}}\cdot\left(\frac{4}{3}dx\right)^2 =$$

$$= \left(\frac{13}{5} + \frac{88}{15}\right)e^{\frac{6}{5}}dx^2 - \frac{\cancel{3}}{5}\cdot\frac{16}{\cancel{9}_3}e^{\frac{6}{5}}dx^2 =$$

$$= \left(\frac{13}{5} + \frac{88}{15} - \frac{16}{15}\right)e^{\frac{6}{5}}dx^2 = \left(\frac{88-16+39}{15}\right)e^{\frac{6}{5}}dx^2$$

$$= \frac{111}{15}e^{\frac{6}{5}}dx^2 > 0$$

$$z_{max} = z\Big|_{P_1, P_2} = e^{-\frac{6}{5}}$$

$$z_{min} = z\Big|_{P_3, P_4} = e^{\frac{6}{5}}$$