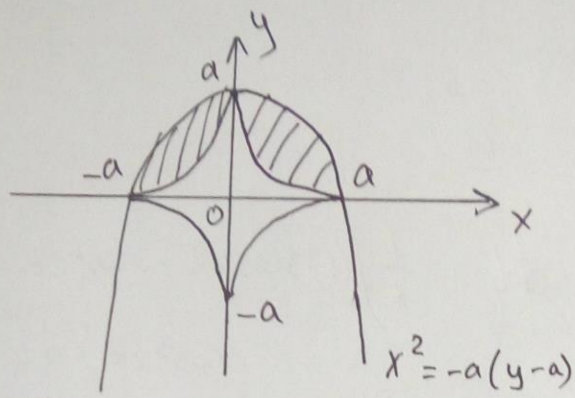
ДЗ №1. Вариант 12.

№1. $\begin{cases} x = a\cos^3 t \\ y = a\sin^3 t \end{cases}$ $x^2 = -a(y-a)$



$y = -\frac{x^2}{a} + a$

т. ∩ с осью Ox:

$-\frac{x^2}{a} + a = 0$

$x^2 = a^2 \quad x = \pm a.$

x/t

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{\pi}{4}$
x	a	0	$-a$	0	a	$\frac{\sqrt{2}}{4}a$
y	0	a	0	$-a$	0	$\frac{\sqrt{2}}{4}a$

$$S_1 = \int_{-a}^{a} y(x)\,dx = -\frac{1}{a}\int_{-a}^{a} x^2 dx + a\int_{-a}^{a} dx =$$

$$= -\frac{1}{a}\cdot\frac{x^3}{3}\Big|_{-a}^{a} + ax\Big|_{-a}^{a} = -\frac{2a^2}{3} + 2a^2 = \frac{4a^2}{3}$$

$$S_2 = 2\int_{\pi/2}^{0} a\sin^3 t \cdot 3a\cos^2 t\,(-\sin t)\,dt =$$

$$= 6a^2 \int_0^{\pi/2} \sin^4 t \cdot \cos^2 t\,dt =$$

$$\int_0^{\pi/2} \sin^4 t\cos^2 t\,dt = \int_0^{\pi/2} \sin^4 t\,dt - \int_0^{\pi/2} \sin^6 t\,dt =$$

$$= \frac{3\pi}{16} - \frac{5\pi}{32}$$

$$\int_0^{\pi/2} \sin^4 t\,dt = \frac{1}{4}\int_0^{\pi/2}(1-\cos 2t)^2\,dt = \frac{1}{4}\int_0^{\pi/2} dt - \frac{1}{4}\int_0^{\pi/2} 2\cos 2t\,dt +$$

$$+\frac{1}{4}\int_0^{\pi/2}\cos^2 2t\,dt = \frac{\pi}{8} - \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{2}\int_0^{\pi/2}\cos 4t\,d(4t) +$$

$$+\frac{1}{2}\cdot\frac{1}{4}\int_0^{\pi/2} dt = \frac{\pi}{8} + \frac{\pi}{16} = \frac{3\pi}{16}$$

$$\int_0^{\pi/2}\sin^6 t\,dt = \frac{1}{8}\int_0^{\pi/2}(1-\cos(2t))^3\,dt = \frac{1}{8}\int_0^{\pi/2}(1-3\cos 2t+3\cos^2 2t -$$

$$-\cos^3 2t)\,dt =$$

$$= \frac{1}{8}\int_0^{\pi/2} dt - \overset{\to 0}{\cancel{\frac{3}{16}\int_0^{\pi/2}\cos 2t\,d(2t)}} + \frac{3}{8}\int_0^{\pi/2}\cos^2 2t\,dt -$$

$$-\frac{1}{8}\int_0^{\pi/2}\cos^3 2t\,dt = \frac{\pi}{16} + \frac{3}{16}\int_0^{\pi/2}(\cos 4t + 1)\,dt -$$

$$-\frac{1}{16}\int_0^{\pi/2}\cos 2t(\cos 4t+1)\,dt = \frac{\pi}{16} + \frac{3\pi}{32} - \frac{1}{16}\int_0^{\pi/2}\cos 2t\cos 4t\,dt -$$

$$-\overset{\to 0}{\cancel{\frac{1}{16}\int_0^{\pi/2}\cos 2t\,dt}} = \frac{5\pi}{32} - \overset{\to 0}{\cancel{\frac{1}{32}\int_0^{\pi/2}\cos 6t\,dt}} - \overset{\to 0}{\cancel{\frac{1}{32}\int_0^{\pi/2}\cos 2t\,dt}} =$$

$$= \frac{5\pi}{32}$$

$$= \; 6a^2\left(\frac{3\pi}{16} - \frac{5\pi}{32}\right) = \frac{6a^2\pi}{32} = \frac{3a^2\pi}{16}$$

$$S_{иском} = S_1 - S_2 = \frac{4a^2}{3} - \frac{3a^2\pi}{16} = \frac{64a^2 - 9\pi a^2}{3\cdot 16} =$$

$$= \frac{a^2}{48}(64-9\pi)$$

Ответ: $\frac{a^2}{48}(64-9\pi)$

№2. $y = e^x$; $y = 1 + 2e^{-x}$, $x = 0$

$V_{oy} - ?$

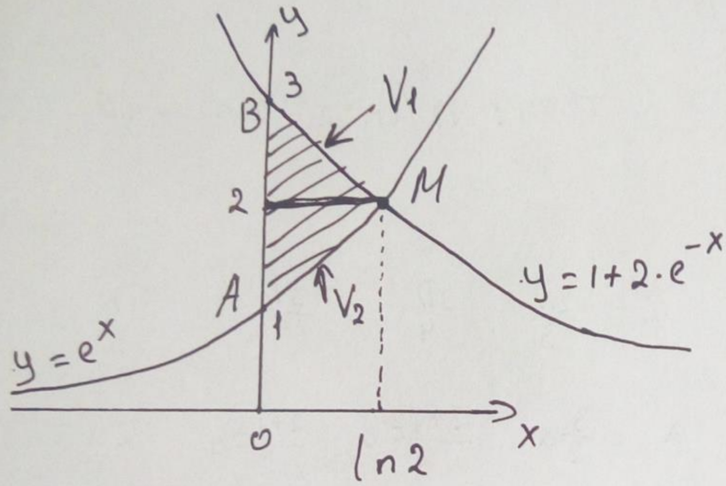
$x = \ln y$;

$x = \ln\frac{2}{y-1}$

Найдём т. пересеч.: $e^x = 1 + 2e^{-x}$ $\quad e^{2x} - e^x - 2 = 0$ $\quad e^x = t$

$t^2 - t - 2 = 0$ $\quad \left[\begin{array}{l} t = 2 \\ t = -1, \ \varnothing, \text{ т.к. } e^x > 0 \end{array}\right.$ $\quad M(2; \ln 2)$;

$y = e^0 = 1$ $\ A(1;0)$; $y = 1 + 2\cdot e^0 = 3$ $\ B(3;0)$



$V_{общ.} = V_1 + V_2$

$$V_1 = \pi \int_2^3 x^2(y)\,dy =$$

$$= \pi \int_2^3 \left(\ln 2 - \ln(y-1)\right)^2 dy =$$

$$= \pi \ln^2 2 \int_2^3 dy - 2\pi\ln 2 \int_2^3 \ln(y-1)\,d(y-1) + \pi \int_2^3 \ln^2(y-1)\,d(y-1) =$$

$$= \pi\ln^2 2 - 2\pi\ln 2\,(y-1)\ln(y-1)\Big|_2^3 + 2\pi\ln 2 \int_2^3 \frac{(y-1)\,d(y)}{(y-1)} +$$

$$+ \pi\ln^2(y-1)(y-1)\Big|_2^3 - 2\pi \int_2^3 \frac{\ln(y-1)(y-1)}{(y-1)}\,d(y-1) =$$

$$= \underline{\pi\ln^2 2} - \underline{4\pi\ln^2 2} + 2\pi\ln 2 + \underline{2\pi\ln^2 2} - 2\pi\ln(y-1)(y-1)\Big|_2^3 +$$

$$+ 2\pi \int_2^3 \frac{(y-1)\,d(y)}{(y-1)} = -\pi\ln^2 2 + 2\pi\ln 2 - 4\pi\ln 2 + 2\pi =$$

$$= -\pi\left(\ln^2 2 - 2\ln 2 + 2\right)$$

$V_2 = \pi\int_1^2 x^2(y)\,dy = \pi\int_1^2 \ln^2 y\,dy = \pi \ln^2 y\cdot y\Big|_1^2 - 2\pi\int_1^2 \frac{y\ln y\,dy}{y} =$

$= 2\pi\ln^2 2 - 2\pi y\ln y\Big|_1^2 - 2\pi\int_1^2\frac{y\,dy}{y} = 2\pi\ln^2 2 - 4\pi\ln 2 - 2\pi =$

$= \pi(2\ln^2 2 - 4\ln 2 - 2)$

$V_{общ.} = V_1 + V_2 = -\pi(\ln^2 2 - 2\ln 2 + 2) + \pi(2\ln^2 2 - 4\ln 2 - 2) =$

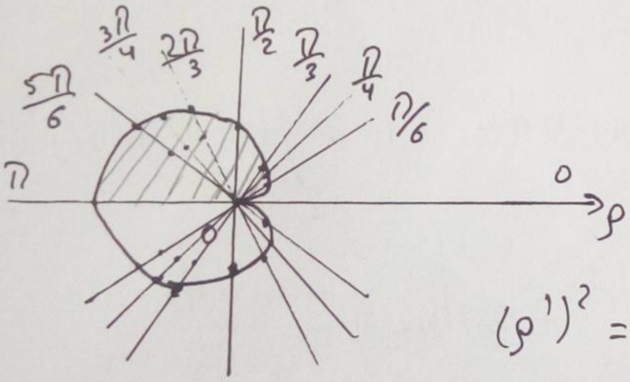
$= \pi\ln^2 2 - 2\pi\ln 2 - 4\pi$

Ответ: $\pi(\ln^2 2 - 2\ln 2 - 4)$

№3. $\rho = a(1-\cos\varphi)$

$Q_{o\rho}$ – ?

φ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
ρ	0	$\frac{2-\sqrt3}{2}a$	$\frac{2-\sqrt2}{2}a$	$\frac12 a$	a	$\frac32 a$	$\frac{2+\sqrt2}{2}a$	$\frac{2+\sqrt3}{2}a$	$2a$



$\rho^2 = a^2(1-\cos\varphi)^2 = a^2 - 2a^2\cos\varphi + a^2\cos^2\varphi$

$\rho' = a\sin\varphi$

$(\rho')^2 = a^2\sin^2\varphi$

$\rho^2 + (\rho')^2 = a^2(1-2\cos\varphi) + a^2 = 2a^2(1-\cos\varphi)$

$Q_{o\rho} = 2\pi\int_0^\pi a(1-\cos\varphi)\sin\varphi\sqrt{2a^2(1-\cos\varphi)}\,d\varphi =$

$= 2\pi\int_0^\pi a^2\sqrt2\,(1-\cos\varphi)^{3/2}\,d(1-\cos\varphi) = 2\sqrt2\,a^2\pi\int_0^\pi(1-\cos\varphi)^{3/2}\,d(1-\cos\varphi) =$

$= \frac{2\sqrt2\,a^2\pi\,(1-\cos\varphi)^{5/2}}{5}\Big|_0^\pi = \frac{16a^2\pi}{5}$ Ответ: $\frac{16a^2\pi}{5}$

№4. $\displaystyle\int_1^{+\infty} \frac{x+\sin x}{x^3(x-\sin x)}\,dx$

$$\frac{x+\sin x}{x^3(x-\sin x)} = \frac{1+\frac{\sin x}{x}}{x^3\left(1-\frac{\sin x}{x}\right)} \sim \frac{1}{x^3} = g(x) \text{ при } x\to+\infty$$

$\displaystyle\int_1^{+\infty} g(x)\,dx = \int_1^{+\infty}\frac{dx}{x^3}$ интеграл Дирихле $\alpha=3>1$, след.

$\displaystyle\int_1^{+\infty} g(x)\,dx = \int_1^{+\infty}\frac{dx}{x^3}$ сходится, тогда

$\displaystyle\int_1^{+\infty}\frac{x+\sin x}{x^3(x-\sin x)}\,dx$ сх-ся по предельному признаку

Ответ: сходится.

№5. $\displaystyle\int_e^{e^2}\frac{x\ln x}{(x-e)^2}\,dx = \Big|$ несоб. интеграл 2-го рода в т. $x=e$ $\Big|$

$$\frac{x\ln x}{(x-e)^2} \geq \frac{e}{(x-e)^2} = g(x)$$

$\displaystyle\int_e^{e^2} g(x)\,dx = e\int_e^{e^2}\frac{dx}{(x-e)^2}$ $\alpha=2>1 \Rightarrow$ расх-ся, тогда

$\displaystyle\int_e^{e^2}\frac{x\ln x}{(x-e)^2}\,dx$ расх-ся по неравенству

Ответ: расходится.