Часть Б

(8) На поверхности $\frac{27}{x}+\frac{8}{y}+\frac{8}{z}=1$ найти точку наименее удалённую от точки $O(0,0,0)$.

Решение:

$\rho=\sqrt{x^2+y^2+z^2}$ наименьшее при условии

$$\frac{27}{x}+\frac{8}{y}+\frac{8}{z}=1$$

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Функция Лагранжа

$$F(x,y,\lambda)=x^2+y^2+z^2+\lambda\left(\frac{27}{x}+\frac{8}{y}+\frac{8}{z}-1\right)$$

$$\begin{cases} F'_x=2x-\dfrac{27\lambda}{x^2}=0, & \Big|\cdot y \quad \Big|\cdot z \\ F'_y=2y-\dfrac{8\lambda}{y^2}=0, & \Big|\cdot x \quad \text{"}-\text{"} \\ F'_z=2z-\dfrac{8\lambda}{z^2}=0, & \quad\quad \Big|\cdot x \quad \text{"}-\text{"} \\ F'_\lambda=\dfrac{27}{x}+\dfrac{8}{y}+\dfrac{8}{z}-1=0 \end{cases}$$

$$\frac{27\lambda y}{x^2}=\frac{8\lambda x}{y^2}$$

$$8\cancel{\lambda}x^3=27\cancel{\lambda}y^3$$

$$y^3=\frac{8}{27}x^3$$

$$\left(y-\frac{2}{3}x\right)\underbrace{\left(y^2+\frac{2}{3}xy+\frac{4}{9}x^2\right)}_{\neq 0}=0$$

$$y=\frac{2}{3}x$$

$$\frac{27\lambda z}{x^2}=\frac{8\lambda x}{z^2}$$

$$8\cancel{\lambda}x^3=27\cancel{\lambda}z^2$$

$$z^3=\frac{8}{27}x^3$$

$$\left(z-\frac{2}{3}x\right)\underbrace{\left(z^2+\frac{2}{3}xz+\frac{4}{9}x^2\right)}_{\neq 0}=0$$

$$z=\frac{2}{3}x$$

$$\frac{12+12+27}{x} = \frac{24+27}{x} = \frac{51}{x} = 1 \Rightarrow x = 51$$

$$y = z = \frac{2}{3} \cdot 51 = 17 \cdot 2 = 34 = y = z$$

$$L = \begin{bmatrix} 0 & \varphi'_x & \varphi'_y & \varphi'_z \\ \varphi'_x & F''_{xx} & F''_{xy} & F''_{xz} \\ \varphi'_y & F''_{xy} & F''_{yy} & F''_{yz} \\ \varphi'_z & F''_{xz} & F''_{yz} & F''_{zz} \end{bmatrix}$$

$\boxed{M(51, 34, 34)}$

$$\varphi'_x = -\frac{27}{x^2}\Big|_M = -\frac{7}{867}$$

$$\varphi'_y = -\frac{8}{y^2}\Big|_M = -\frac{2}{289}$$

$$\varphi'_z = -\frac{8}{z^2}\Big|_M = -\frac{2}{289}$$

Найдём λ: $\frac{27\lambda}{x^2} = 2x$

$$27\lambda = 2x^3$$

$$\lambda = \frac{2x^3}{27} = \frac{2 \cdot \overset{17}{\cancel{51}} \cdot \overset{17}{\cancel{51}} \cdot \overset{17}{\cancel{51}}}{\cancel{27}} = 2 \cdot 17^3 =$$

$$= 9826$$

$$F''_{xx} = 2 + 2 \cdot 27\lambda \cdot \frac{1}{x^3} = 2 + 2 \cdot \cancel{27} \cdot 2 \cdot \cancel{17} \cdot \cancel{17} \cdot \cancel{17} \cdot \frac{1}{\cancel{51^3}_{3^3}} = 6$$

$$F''_{yy} = 2 + 2 \cdot 8\lambda \cdot \frac{1}{y^3} = 2 + 2 \cdot \cancel{8} \cdot 2 \cdot \cancel{17} \cdot \cancel{17} \cdot \cancel{17} \cdot \frac{1}{\cancel{34^3}_{2^3}} = 6$$

$$F''_{zz} = 2 + 2 \cdot 8\lambda \cdot \frac{1}{z^3} = 2 + 2 \cdot \cancel{8} \cdot 2 \cdot \cancel{17} \cdot \cancel{17} \cdot \cancel{17} \cdot \frac{1}{\cancel{34^3}_{2^3}} = 6$$

$$L=\begin{bmatrix} 0 & -\frac{7}{867} & -\frac{2}{289} & -\frac{2}{289} \\ -\frac{7}{867} & 6 & 0 & 0 \\ -\frac{2}{289} & 0 & 6 & 0 \\ -\frac{2}{289} & 0 & 0 & 6 \end{bmatrix}$$

$F''_{xy}=0$

$F''_{yz}=0$

$F''_{xz}=0.$

$$\det L = -\frac{2}{289}M_1 - 6M_2$$

$$M_1=\begin{vmatrix} -\frac{7}{867} & -\frac{2}{289} & -\frac{2}{289} \\ 6 & 0 & 0 \\ 0 & 6 & 0 \end{vmatrix} = -\frac{2\cdot 36}{289}$$

$$M_2=\begin{vmatrix} 0 & -\frac{7}{867} & -\frac{2}{289} \\ -\frac{7}{867} & 6 & 0 \\ -\frac{2}{289} & 0 & 6 \end{vmatrix} = -\frac{2\cdot 2\cdot 6}{289^2}$$

$\det L = \frac{2^2\cdot 6^2}{289^2}+\frac{2^2\cdot 6^2}{289^2} = 2\cdot\frac{2^2\cdot 6^2}{289^2} > 0 \Rightarrow$ условный минимум

$\Rightarrow$ в точке $M(51,34,34)$ ф-ция S достигает своего наименьшего значения.

$$\rho = \sqrt{51^2 + 34^2 + 34^2} = \sqrt{17^2 \cdot 3^2 + 17^2 \cdot 2^2 + 17^2 \cdot 2^2} =$$

$$= 17 \cdot \sqrt{9+4+4} = 17\sqrt{17}$$

Ответ: $M(51, 34, 34)$