Приложение квадратичных форм. В 12.

а) $x^2+y^2-2xy+5\sqrt{2}\,x+\sqrt{2}\,y+8=0$

Матрица кв. формы: $A=\begin{pmatrix}1&-1\\-1&1\end{pmatrix}$

$\det(A-\lambda E)=0 \qquad (1-\lambda)^2-1=0 \qquad \lambda(\lambda-2)=0$

$\lambda_1=0;\ \lambda_2=2$ - собств. числа

$\cancel{\begin{pmatrix}1&-1\\-1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=0} \Rightarrow \left\{\cancel{x_1-x_2=0}\right.$

$\underline{\lambda_1=0}$

$\begin{pmatrix}1&-1\\-1&1\end{pmatrix}\begin{pmatrix}x_1\\y_1\end{pmatrix}=0 \Rightarrow \begin{cases}x_1-y_1=0\\-x_1+y_1=0\end{cases}$

$Rg=1 \Rightarrow x_1$ - б. пер.; y_1 - св. пер.

$\begin{cases}y_1=c\\x_1=c\end{cases} \qquad \cancel{E_1=(1,1)^T} \quad E_1=\begin{pmatrix}x_1\\y_1\end{pmatrix}=c\begin{pmatrix}1\\1\end{pmatrix}$

$c=1 \Rightarrow E_1=\begin{pmatrix}1\\1\end{pmatrix} \qquad \bar{e}_1=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

$\underline{\lambda_2=2}$

$\begin{pmatrix}-1&-1\\-1&-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=0 \qquad x+y=0$

$Rg=1 \Rightarrow x$ - б. пер.; y - св. пер.

$\begin{cases}y=c\\x=-c\end{cases} \qquad X^{\lambda=2}=\begin{pmatrix}x\\y\end{pmatrix}=c\begin{pmatrix}-1\\1\end{pmatrix} \qquad c=1 \Rightarrow$

$E_2=\begin{pmatrix}-1\\1\end{pmatrix}$

$\bar{e}_2=\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}$

$U=\frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\1&1\end{pmatrix}; \qquad \begin{cases}x=\frac{1}{\sqrt{2}}x'-\frac{1}{\sqrt{2}}y'\\ y=\frac{1}{\sqrt{2}}x'+\frac{1}{\sqrt{2}}y'\end{cases}$

$$\begin{cases} x = \frac{1}{\sqrt{2}}(x'-y') \\ y = \frac{1}{\sqrt{2}}(x'+y') \end{cases}$$ подставим в $x^2+y^2-2xy+5\sqrt{2}\,x+\sqrt{2}\,y+8=0$

$$\frac{1}{2}(x'-y')^2+\frac{1}{2}(x'+y')^2-2\cdot\frac{1}{2}(x'^2-y'^2)+$$

$$+5\sqrt{2}\cdot\frac{1}{\sqrt{2}}(x'-y')+\frac{\sqrt{2}}{\sqrt{2}}(x'+y')+8=0$$

$$=\frac{1}{2}\left(x'^2-2x'y'+y'^2+x'^2+2x'y'+y'^2\right)+y'^2-x'^2+$$

$$+5x'-5y'+x'+y'+8=x'^2+y'^2+y'^2-x'^2+6x'-4y'+8=$$

$$=2y'^2-4y'+8+6x'=0$$

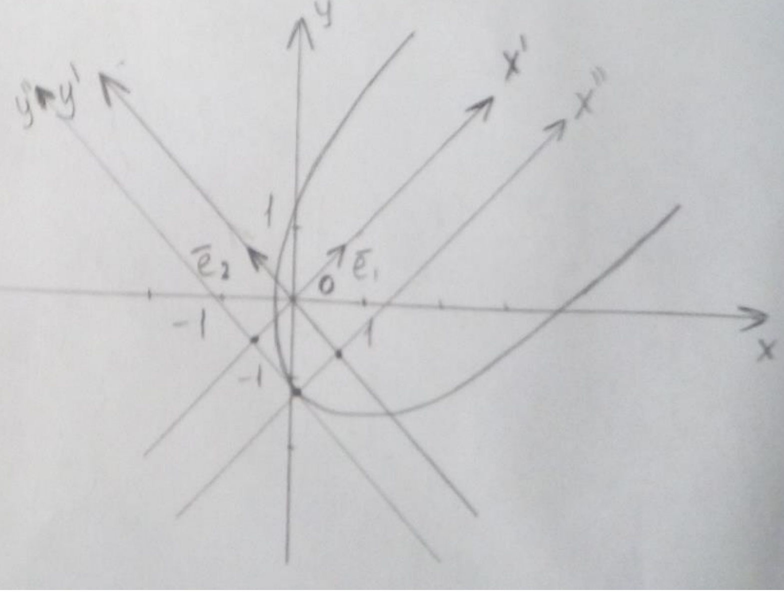
$$y'^2-2y'+4=3x' \qquad (y'-1)^2=3x'-2$$ – ур-е параболы

переход в канонич. систему координат:

$$(y'-1)^2=3\left(x'-\frac{2}{3}\right)$$

$$\begin{cases} y''=y'-1 \\ x''=x'-\frac{2}{3} \end{cases} \qquad y''^2=3x'' \qquad 3=2p$$

$$\begin{cases} x=\frac{1}{\sqrt{2}}\left(x''+\frac{2}{3}-y''-1\right)=\frac{1}{\sqrt{2}}\left(x''-y''-\frac{1}{3}\right) \\ y=\frac{1}{\sqrt{2}}\left(x''+y''+\frac{5}{3}\right) \end{cases}$$



б) $6xy - 5x^2 + 3y^2 - 8\sqrt{10}\,x + 4 = 0$

Матрица кв. формы: $A = \begin{pmatrix} -5 & 3 \\ 3 & 3 \end{pmatrix}$

$\det(A - \lambda E) = 0$

$(-5-\lambda)(3-\lambda) - 9 = 0 \qquad -15 - 3\lambda + 5\lambda + \lambda^2 - 9 = 0$

$\lambda^2 + 2\lambda - 24 = 0 \qquad \lambda_1 = -6;\ \lambda_2 = 4$ – собств. числа.

$\underline{\lambda_1 = -6}$

$\begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow \quad x + 3y = 0$

$Rg = 1 \Rightarrow x$ – б. пер.; y – св. пер.

$\begin{cases} y = c \\ x = -3c \end{cases} \qquad X^{\lambda_1=-6} = \begin{pmatrix} x \\ y \end{pmatrix} = c\begin{pmatrix} -3 \\ 1 \end{pmatrix} \qquad c = 1 \Rightarrow$

$E_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$\bar{e}_1 = \frac{1}{\sqrt{10}}(-3, 1)^T$

$\underline{\lambda_2 = 4}$

$\begin{pmatrix} -9 & 3 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow \quad -9x + 3y = 0$

$y = 3x$

$Rg = 1 \Rightarrow x$ – б. пер.; y – св. пер.

$\begin{cases} y = c \\ x = \frac{1}{3}c \end{cases} \qquad X^{\lambda_2=4} = \begin{pmatrix} x \\ y \end{pmatrix} = c\begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \qquad c = 3 \Rightarrow E_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\bar{e}_2 = \frac{1}{\sqrt{10}}(1, 3)^T$

$U = \frac{1}{\sqrt{10}}\begin{pmatrix} -3 & 1 \\ 1 & 3 \end{pmatrix} \qquad \begin{cases} x = \frac{1}{\sqrt{10}}(-3x' + y') \\ y = \frac{1}{\sqrt{10}}(x' + 3y') \end{cases}$

подставим в $\quad 6xy - 5x^2 + 3y^2 - 8\sqrt{10}\,x + 4 = 0$

$$\frac{6}{10}(-3x'+y')(x'+3y') - \frac{5}{10}(-3x'+y')^2 + \frac{3}{10}(x'+3y')^2 -$$

$$\cancel{-18} - 8(-3x'+y') + 8 = 0$$

$$= \frac{6}{10}\left(-3x'^2 + x'y' - 9x'y' + 3y'^2\right) - \frac{5}{10}\left(9x'^2 - 6x'y' + y'^2\right) + \frac{3}{10}(x'+3y')^2$$

$$- 8(-3x'+y') + 8 = \frac{-18x'^2 - 48x'y' + 18y'^2 - 45x'^2 + 30x'y' - 5y'^2}{10}$$

$$+ \frac{3x'^2 + 18x'y' + 27y'^2}{10} - 8(-3x'+y') + 8 =$$

$$\cancel{= \frac{-60x'^2 - 60x'y' + 50y'^2}{10} + 24x' - 8y' + 8 =}$$

$$\cancel{= -6x'^2 - 6x'y' + 5y'^2 + 24x' - 8y' + 8 = 0}$$

$$= \frac{-60x'^2 + 40y'^2}{10} + 24x' - 8y' + 8 = -6x'^2 + 4y'^2 + 24x' - 8y' + 8 = 0$$

$$\cancel{2y} \quad \cancel{3x'^2 - 12x' - 8} \quad 3x'^2 - 12x' - 2y'^2 + 4y' - 4 = 0$$

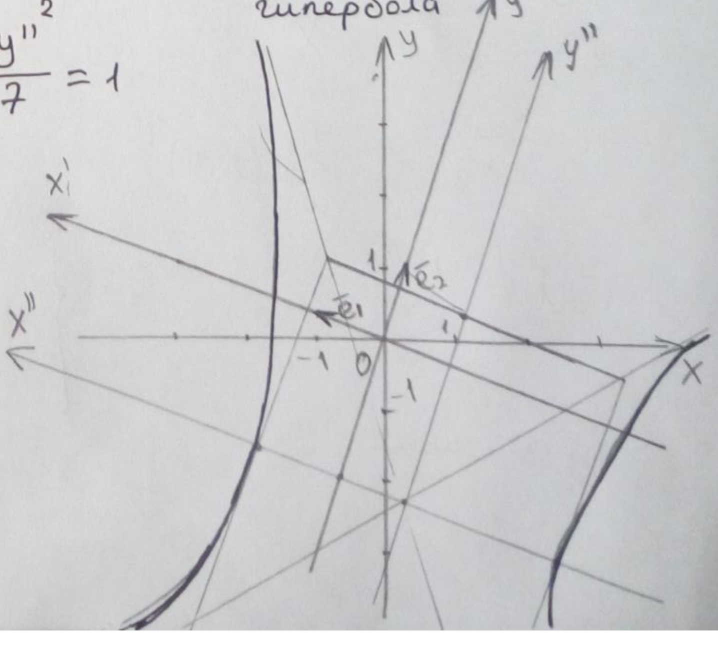
$$3(x'^2 - 4x' + 4) - 12 - 2(y'^2 - 2y' + 1) + 2 - 4 = 0$$

$$3(x'-2)^2 - 2(y'-1)^2 = 14 \qquad \frac{(x'-2)^2}{14/3} - \frac{(y'-1)^2}{7} = 1$$

гипербола

$$\begin{cases} x'' = x' - 2 \\ y'' = y' - 1 \end{cases} \qquad \frac{x''^2}{\frac{14}{3}} - \frac{y''^2}{7} = 1$$

$$\begin{cases} x = \frac{1}{\sqrt{10}}(-3x'' + y'' - 5) \\ y = \frac{1}{\sqrt{10}}(x'' + 3y'' + 5) \end{cases}$$



$\sqrt{\frac{14}{3}} \simeq 2,2 \quad \sqrt{7} \simeq 2,7.$

8) c) $7x^2+6y^2+5z^2-4xy-4yz-6x-\cancel{2y}\,24y+18z-72=0.$

Матрица кв. формы: $A=\begin{pmatrix}7&-2&0\\-2&6&-2\\0&-2&5\end{pmatrix}$

$\det(A-\lambda E)=0$

$$\begin{vmatrix}7-\lambda&-2&0\\-2&6-\lambda&-2\\0&-2&5-\lambda\end{vmatrix}=0$$

$$(7-\lambda)\begin{vmatrix}6-\lambda&-2\\-2&5-\lambda\end{vmatrix}+2\begin{vmatrix}-2&-2\\0&5-\lambda\end{vmatrix}=$$

$$=(7-\lambda)\left((6-\lambda)(5-\lambda)-4\right)-4(5-\lambda)=$$

$$=(7-\lambda)\left(30-5\lambda-6\lambda+\lambda^2-4\right)-20+4\lambda=$$

$$=\cancel{7\cdot\lambda}\ 7\lambda^2-77\lambda+7\cdot26-\lambda^3+11\lambda^2-26\lambda-20+4\lambda=$$

$$=-\lambda^3+18\lambda^2-99\lambda+162=0 \qquad \lambda_1=3$$

$$\begin{array}{l|l}
-\lambda^3+18\lambda^2-99\lambda+162 & \lambda-3\\
\hline
-\lambda^3+3\lambda^2 & -\lambda^2+15\lambda-54\\
\quad 15\lambda^2-99\lambda & \\
\quad 15\lambda^2-45\lambda & \\
\qquad -54\lambda+162 & \\
\qquad -54\lambda+162 & \\
\qquad\qquad 0 &
\end{array}$$

$$(\lambda-3)(-\lambda^2+15\lambda-54)=0$$

$$\left[\begin{array}{l}\lambda_1=3\\ \lambda_2=6\\ \lambda_3=9\end{array}\right.$$ – собств. числа.

$\underline{\lambda_1=3}$

$$\begin{pmatrix}4&-2&0\\-2&3&-2\\0&-2&2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=0 \qquad \begin{cases}4x-2y=0\\-2x+3y-2z=0\\-2y+2z=0.\end{cases}$$

$\cancel{\Leftrightarrow Rg=3}$

$\cancel{-2\ -y+8y-2y=0}$

$$\begin{cases}2x=y\\y=z\\-2x+3y-2z=0\end{cases} \qquad X^{\lambda_1=3}=\begin{pmatrix}x\\y\\z\end{pmatrix}=c\begin{pmatrix}1\\2\\2\end{pmatrix} \quad c=1 \Rightarrow$$

$$E_1=(1;2;2)^T \qquad \bar{e}=\frac{1}{3}(1;2;2)^T$$

$\lambda_2 = 6$

$$\begin{pmatrix} 1 & -2 & 0 \\ -2 & 0 & -2 \\ 0 & -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \qquad \begin{cases} x - 2y = 0 \\ x = -z \\ -2y = z \end{cases} \qquad \cancel{x = 2y} \quad \cancel{x =}$$

$$\begin{cases} y = \dfrac{x}{2} \\ z = -x \\ z = -2y \end{cases} \qquad X^{\lambda_2 = 6} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = c \begin{pmatrix} 1 \\ \frac{1}{2} \\ -1 \end{pmatrix} \quad c = 2 \Rightarrow$$

$$E_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \qquad \bar{e}_2 = \frac{1}{3}(2; 1; -2)^T$$

$\lambda_3 = 9$

$$\begin{pmatrix} -2 & -2 & 0 \\ -2 & -3 & -2 \\ 0 & -2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \qquad \begin{cases} y = -x \\ -2x + 3y + 2z = 0 \\ y = -2z \end{cases} \qquad \begin{cases} x = c \\ y = -c \\ z = \dfrac{c}{2} \end{cases}$$

$$X^{\lambda_3 = 9} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = c \begin{pmatrix} 1 \\ -1 \\ \frac{1}{2} \end{pmatrix} \qquad c = 2 \Rightarrow E_3 = (2; -2; 1)^T$$

$$\bar{e}_3 = \frac{1}{3}(2; -2; 1)^T \qquad U = \frac{1}{3} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$\begin{cases} x = \frac{1}{3}(x' + 2y' + 2z') \\ y = \frac{1}{3}(2x' + y' - 2z') \\ z = \frac{1}{3}(2x' - 2y' + z') \end{cases} \qquad 2b^T U X' = \frac{1}{3}(-6; -24; +18) \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} =$$

$$= \frac{1}{3}(-6; -24; +18) \begin{pmatrix} x' + 2y' + 2z' \\ 2x' + y' - 2z' \\ 2x' - 2y' + z' \end{pmatrix} =$$

$$= \frac{1}{3}(-6x' - 12y' - 12z' - 48x' - 24y' + 48z' + 36x' - 36y' + 18z') =$$

$$= \frac{1}{3}(-18x' - 72y' + 54z') = -6x' - 24y' + 18z'$$

$$3x'^2 + 6y'^2 + 9z'^2 - 6x' - 24y' + 18z' - 72 = 0$$

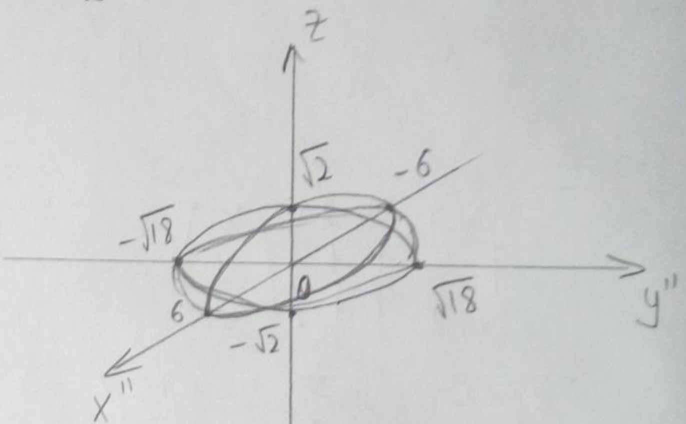
$$3(x'^2 - 2x' + 1) - 3 + 6(y'^2 - 4y' + 4) - 24 + 9(z'^2 + 2z' + 1) - 9 - 72 = 0.$$

$$3(x'-1)^2 + 6(y'-2)^2 + 9(z'+1)^2 = 108$$

$$\frac{(x'-1)^2}{36} + \frac{(y'-2)^2}{18} + \frac{(z'+1)^2}{12} = 1$$ эллипсоид.

$$\begin{cases} x'' = x' - 1 \\ y'' = y' - 2 \\ z'' = z' + 1 \end{cases} \qquad \frac{x''^2}{36} + \frac{y''^2}{18} + \frac{z''^2}{2} = 1$$

$$\begin{cases} x = \frac{1}{3}(x'' + 2y'' + 2z'' + 3) \\ y = \frac{1}{3}(2x'' + y'' - 2z'' + 6) \\ z = \frac{1}{3}(2x'' - 2y'' + z'' - 3) \end{cases}$$



d) $-5x^2 + y^2 + z^2 + 4xy + 4xz + 2yz - 3\sqrt{2}\,y + 3\sqrt{2}\,z + 12 = 0.$

Матрица кв. формы: $A = \begin{pmatrix} -5 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$

$\det(A - \lambda E) = 0$

$$\begin{vmatrix} -5-\lambda & 2 & 2 \\ 2 & 1-\lambda & 1 \\ 2 & 1 & 1-\lambda \end{vmatrix} = 0 \qquad (-5-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 2 & 1-\lambda \end{vmatrix} + 2\begin{vmatrix} 2 & 1-\lambda \\ 2 & 1 \end{vmatrix} =$$

$$= (-5-\lambda)\left((1-\lambda)^2 - 1\right) - 2(2 - 2\lambda - 2) + 2(2 - 2 + 2\lambda) =$$

$$= (-5-\lambda)(\lambda^2 - 2\lambda) \cancel{2} + 4\lambda + 4\lambda = \underline{-5\lambda^2} - \lambda^3 + 10\lambda + \underline{2\lambda^2} + 8\lambda =$$

$$= -\lambda^3 - 3\lambda^2 + 18\lambda = 0 \qquad \lambda^3 + 3\lambda^2 - 18\lambda = \lambda(\lambda^2 + 3\lambda - 18) = 0$$

$\lambda_1 = -6;\ \lambda_2 = 0;\ \lambda_3 = 3$ — собств. числа

$\lambda_1 = -6$

$$\begin{pmatrix} -11 & 2 & 2 \\ 2 & -5 & 1 \\ 2 & 1 & -5 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 2 \\ 2 & 7 & 1 \\ 2 & 1 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \qquad \begin{cases} x + 2y + 2z = 0 \\ 2x + 7y + z = 0 \\ 2x + y + 7z = 0 \end{cases}$$

$$\begin{cases} 6y = 6z \\ x + 2y + 2z = 0 \\ 2x + 7y + z = 0 \end{cases} \qquad \begin{cases} y = z \\ x = -4y \end{cases} \qquad \begin{cases} x = -4c \\ y = c \\ z = c \end{cases}$$

$$X^{\lambda_1 = -6} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = c \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix} \qquad c = 1 \Rightarrow E_1 = (-4; 1; 1)^T$$

$$\bar{e}_1 = \frac{1}{\sqrt{18}}(-4; 1; 1)^T$$

$\lambda_2 = 0$

$$\begin{pmatrix} -5 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \qquad \begin{cases} -5x + 2y + 2z = 0 \\ 2x + y + z = 0 \end{cases}$$

$$\cdot(-2) \quad \begin{cases} -5x + 2y + 2z = 0 \\ -4x - 2y - 2z = 0 \end{cases} \qquad \begin{cases} x = 0 \\ y = -z \end{cases}$$

$$X^{\lambda_2 = 0} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = c \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \qquad c = 1 \Rightarrow E_2 = (0; -1; +1)^T$$

$$\bar{e}_2 = \frac{1}{\sqrt{2}}(0; -1; 1)^T$$

$\lambda_3 = 3$

$$\begin{matrix} -8 & 2 & 2 \\ 2 & -2 & -2 \\ 2 & -2 & -2 \end{matrix} \quad \begin{pmatrix} -8 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \qquad \begin{cases} -8x + 2y + 2z = 0 \\ 2x - 2y + z = 0 \\ 2x + y - 2z = 0 \end{cases}$$

$$\begin{cases} y = z \\ 2x = y \end{cases} \qquad X^{\lambda_3 = 3} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = c \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad c = 1 \Rightarrow E_3 = (1; 2; 2)^T$$

$$\bar{e}_3 = \frac{1}{3}(1; 2; 2)^T$$

$$U = \frac{1}{3\sqrt{2}}\begin{pmatrix} -4 & 0 & \sqrt{2} \\ 1 & -3 & 2\sqrt{2} \\ 1 & 3 & 2\sqrt{2} \end{pmatrix}$$

$$\begin{cases} x = \frac{1}{3\sqrt{2}}\left(-4x' + \sqrt{2}z'\right) \\ y = \frac{1}{3\sqrt{2}}\left(x' - 3y' + 2\sqrt{2}z'\right) \\ z = \frac{1}{3\sqrt{2}}\left(x' + 3y' + 2\sqrt{2}z'\right) \end{cases}$$

$$2b^T U x' = \frac{1}{3\sqrt{2}}\left(0; -3\sqrt{2}; 3\sqrt{2}\right)\begin{pmatrix} -4x' + \sqrt{2}z' \\ x' - 3y' + 2\sqrt{2}z' \\ x' + 3y' + 2\sqrt{2}z' \end{pmatrix} =$$

$$= (0; -1; 1)\begin{pmatrix} -4x' + \sqrt{2}z' \\ x' - 3y' + 2\sqrt{2}z' \\ x' + 3y' + 2\sqrt{2}z' \end{pmatrix} = -x' + 3y' - 2\sqrt{2}z' + x' + 3y' + 2\sqrt{2}z' =$$

$$= 6y'$$

$$-6x'^2 + 3y'^2 + 6y' - 12 = 0 \qquad 3(y'+1)^2 - 3 - 12 - 6x'^2 = 0$$

$$3(y'+1)^2 - 6x'^2 = 15 \qquad \frac{(y'+1)^2}{5} - \frac{x'^2}{5/2} = 1$$ – гиперболич. цилиндр.

$$\begin{cases} x'' = y' + 1 \\ y'' = x' \\ z'' = z' \end{cases} \qquad \boxed{\frac{x''^2}{5} - \frac{y''^2}{5/2} = 1}$$

$$\begin{cases} x = \frac{1}{3\sqrt{2}}\left(-4y'' + \sqrt{2}z''\right) \\ y = \frac{1}{3\sqrt{2}}\left(-3x'' + y'' + 2\sqrt{2}z'' + 3\right) \\ z = \frac{1}{3\sqrt{2}}\left(3x'' + y'' + 2\sqrt{2}z'' - 3\right) \end{cases}$$